On risk and uncertainty, and objective versus subjective probability

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Aims of the paper

1. Use the algebra, $\mathbb{R} \times \mathbb{R}$, to characterize decision makers' representations of risk and uncertainty.

2. Show that:

   objective probabilities represent risk and

   subjective probabilities represent uncertainty

   within different domains of the algebra.

3. Decision makers who use the algebra to represent their beliefs may behave in plausible but surprising ways

Risk versus uncertainty

Risk: a situation in which the sample space is measurable.

Examples: coin tosses, roulette wheels, ‘scientific-experiments’

Uncertainty: a situation in which the sample space is not (wholly) measurable.

Examples: urn draws, horse lotteries, decision problems

F.H. Knight (1921) *Risk, Uncertainty, and Profit*

J.M. Keynes (1921) *A Treatise on Probability*

Objective versus subjective probability

Objective probability: a measure of a sample space and its associated σ–algebra which is *independent of the subject*. (e.g., Frequentist statistics)

Examples: coin tosses, roulette wheels, ‘scientific-experiments’,

Subjective probability: a measure of a sample space and its associated σ–algebra which is *imposed by the subject*. (e.g., Bayesian statistics)

Examples: urn draws, horse lotteries, ‘decision problems’,

1.1 the ring

objective probability

$a + b\acute{e}$ (with: $a, b \in \mathbb{R}$)

subjective probability

$$(a + b\acute{e}) + (c + d\acute{e}) = (a + c) + (b + d)\acute{e}$$

$$(a + b\acute{e}) \cdot (c + d\acute{e}) = (ac) + (ad + bc + bd)\acute{e}$$

$$\acute{E} \cong \mathbb{R} \times \mathbb{R}$$

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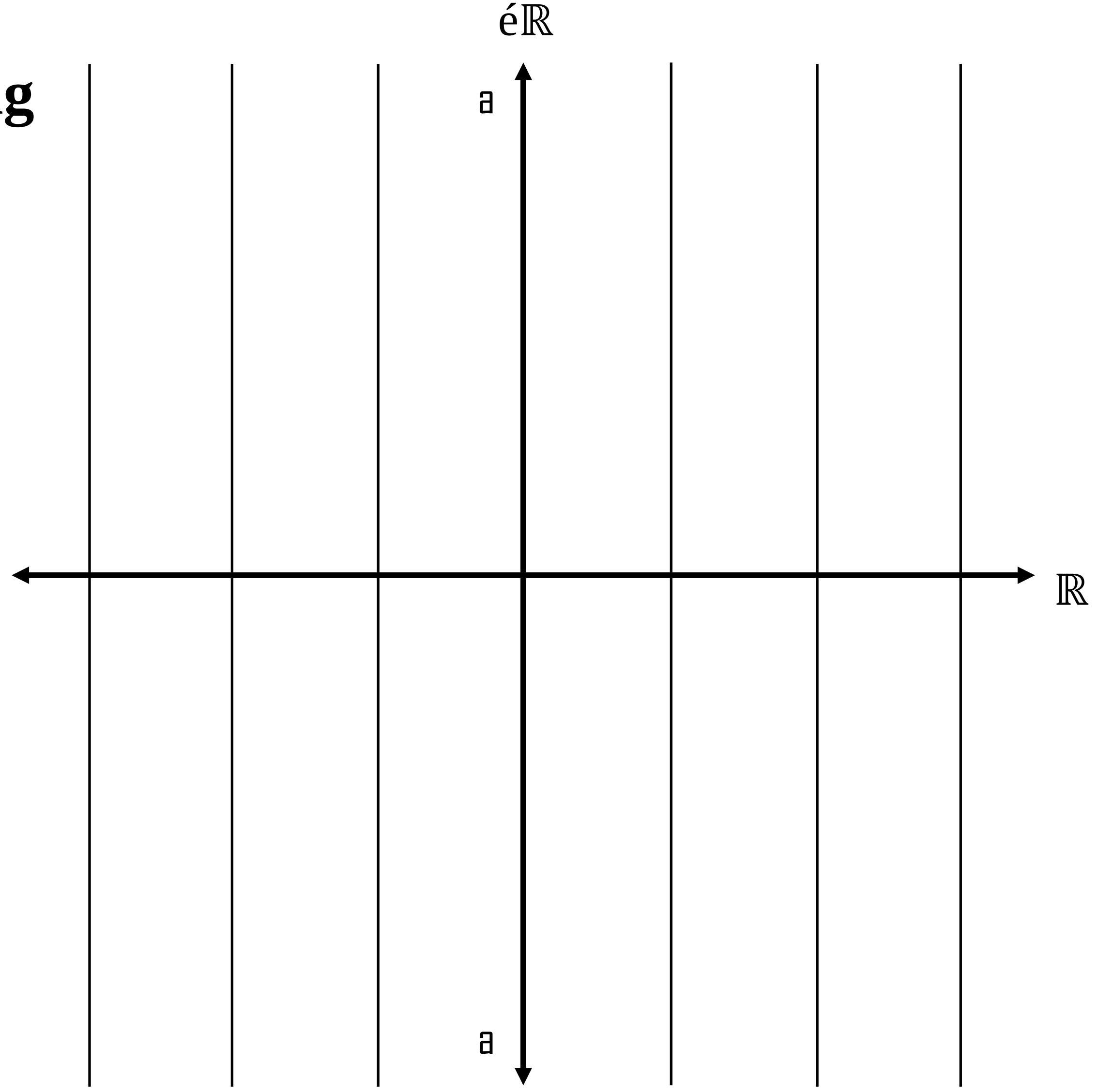
1.2 the ring – in matrix form

$$\mathcal{M}(a + b\acute{e}) = \begin{pmatrix} a & 0 \\ b & a+b \end{pmatrix}$$

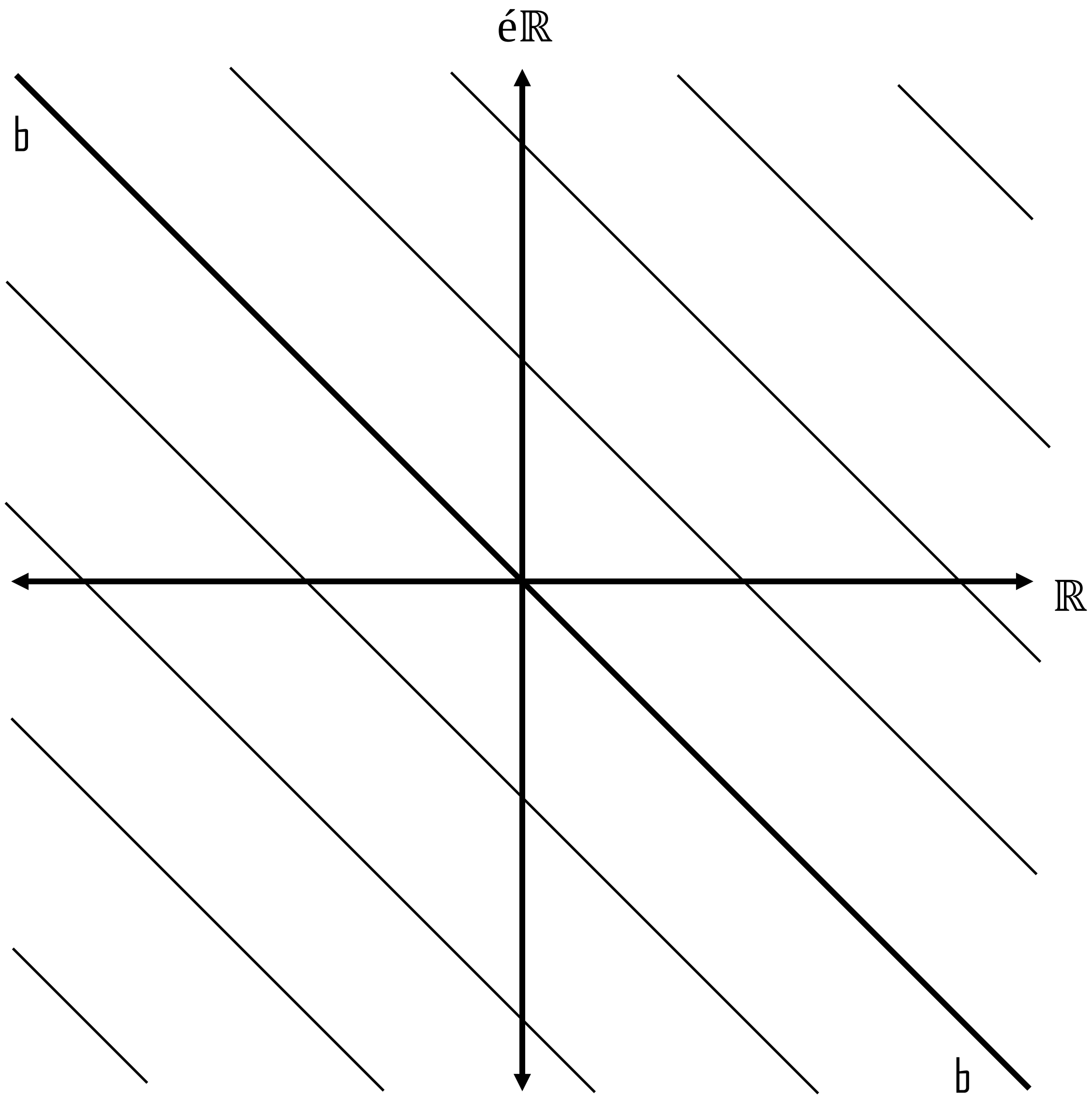
zero/null $\quad \mathcal{M}(0 + 0\acute{e}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathcal{M}(1 + 0\acute{e}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ identity/unity

$\mathcal{M}(0 + 1\acute{e}) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \qquad \mathcal{M}(1 - 1\acute{e}) = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$ idempotents

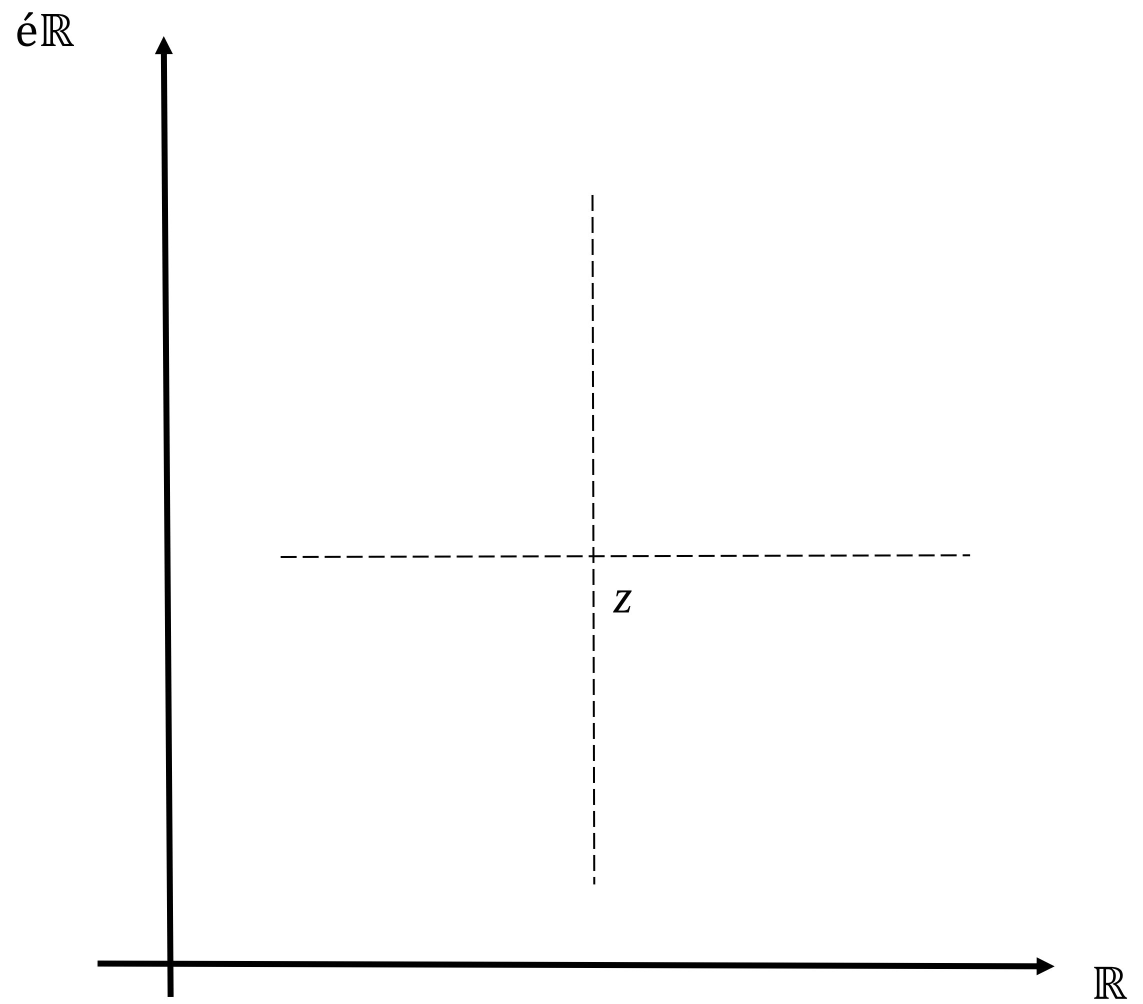
1.3a ideals of the ring



1.3b ideals of the ring



1.4 numbers are partially ordered



1.5 grid addition (and the 1-norm)

$$\lfloor a + b\acute{e} \rfloor = a + b \in \mathbb{R}$$

$$\lfloor a + b\acute{e} \rfloor + \lfloor c + d\acute{e} \rfloor = \lfloor (a + b\acute{e}) + (c + d\acute{e}) \rfloor$$

2.0 beliefs

algebra of events

$\mu_{\acute{e}}: \mathcal{F} \rightarrow \acute{E}:$

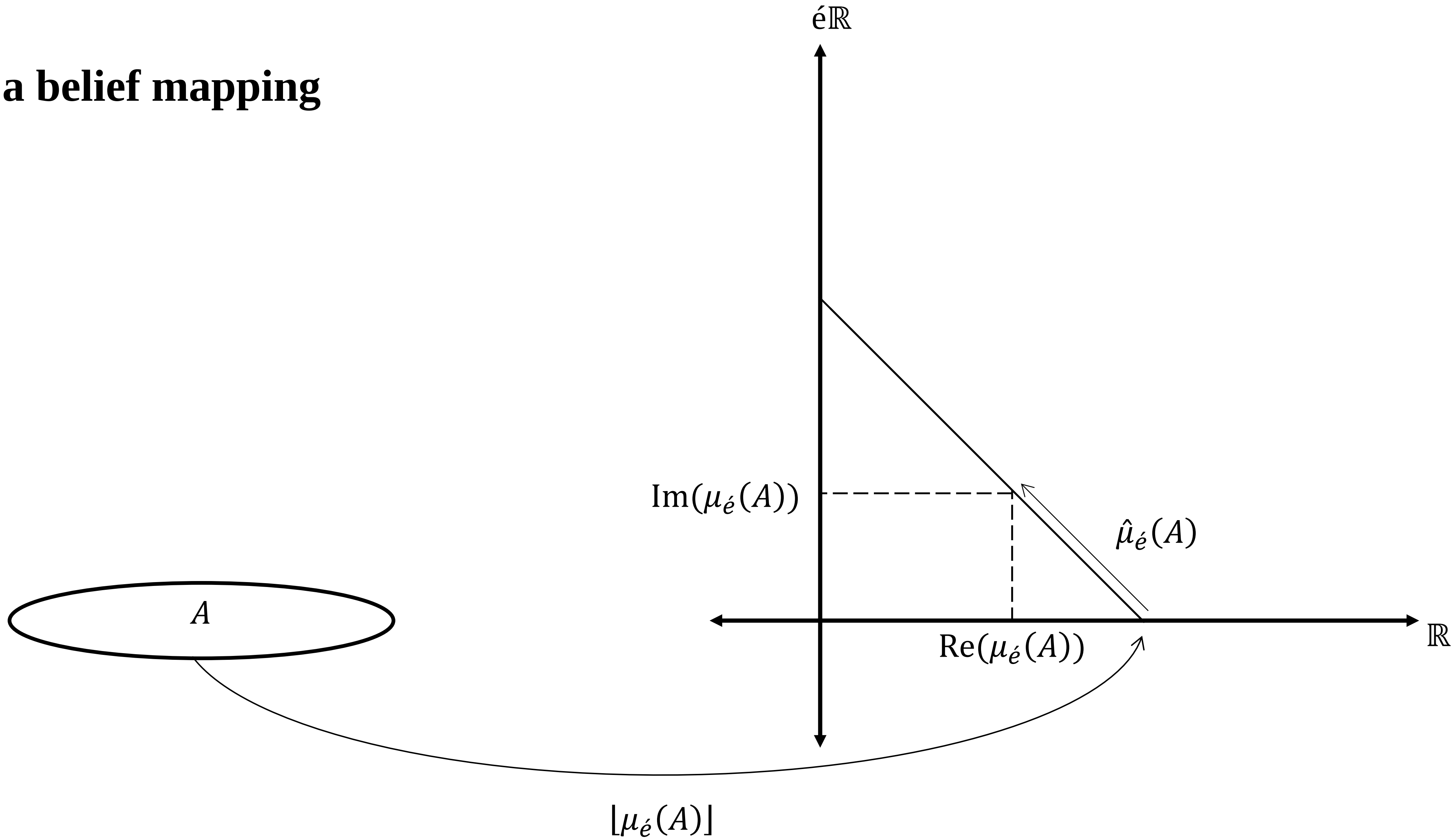
1. $\mu_{\acute{e}}(\Omega) = 1 \quad (\Rightarrow \lfloor \mu_{\acute{e}}(\Omega) \rfloor = 1)$

2. $0 \leq \lfloor \mu_{\acute{e}}(A) \rfloor \leq 1 \quad \forall A \in \mathcal{F}$

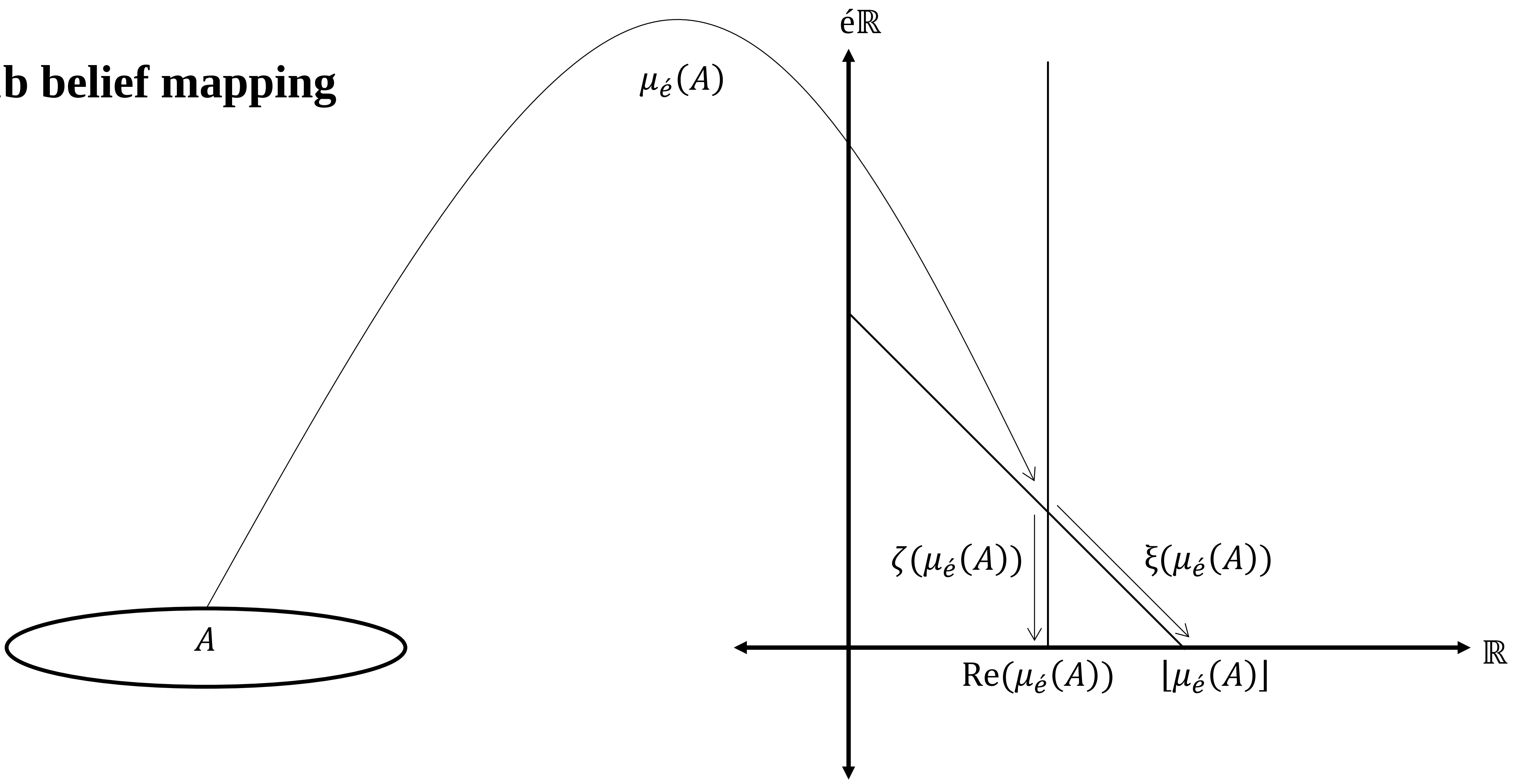
3. $\lfloor \mu_{\acute{e}}(A \cup B) \rfloor = \lfloor \mu_{\acute{e}}(A) \rfloor + \lfloor \mu_{\acute{e}}(B) \rfloor = \lfloor \mu_{\acute{e}}(A) + \mu_{\acute{e}}(B) \rfloor$ when $A \cap B = \emptyset$

4. $0 \leq \mu_{\acute{e}}(A) \quad \forall A \in \mathcal{F}.$

2.1a belief mapping



2.1b belief mapping



ξ = homomorphism whose kernel is $\mathfrak{b}$

ζ= homomorphism whose kernel is $\mathfrak{a}$

2.2 Keynes' 'weight of argument'

$$w \triangleq \frac{\mathrm{Re}(\mu_{\acute{e}}(A))}{\lfloor \mu_{\acute{e}}(A) \rfloor}$$

$w = 1$ for any event implies the probability is completely reliable

$w = 0$ for any event implies the probability is completely unreliable

$w = 1$ for all events implies the situation is one of 'risk'

$w = 0$ for all events implies the situation is one of 'uncertainty'

2.3a Example: the Ellsberg 3 colour problem

$$\mu_{\acute{e}}(b) = 0 + 1/3\,\acute{e} = \mu_{\acute{e}}(y),$$

$$\mu_{\acute{e}}(r) = 1/3 + 0\acute{e}$$

$$\mu_{\acute{e}}(r \cup b) = 1/3 + 1/3\,\acute{e} = \mu_{\acute{e}}(r \cup y)$$

$$\mu_{\acute{e}}(b \cup y) = 2/3 + 0\acute{e}$$

2.3b Example: the boxer, the wrestler and the coin flip*

	Heads	Tails
White	$(0,\frac{1}{4})$	$(0,\frac{1}{4})$
Black	$(0,\frac{1}{4})$	$(0,\frac{1}{4})$

***Andrew Gelman**

2.3b Example: the boxer, the wrestler and the coin flip

$$\mu_{é}(hb) = \mu_{é}(hw) = \mu_{é}(tb) = \mu_{é}(tw) = 0 + 1/4\,é$$

$$\mu_{é}(black) = \mu_{é}(white) = 0 + 1/2\,é$$

$$\mu_{é}(heads) = \mu_{é}(tails) = 1/2 + 0é$$

$$\mu_{é}(hb^{c}) = \mu_{é}(hw^{c}) = \mu_{é}(tb^{c}) = \mu_{é}(tw^{c}) = 1/2 + 1/4\,é$$

3.0 lotteries

The objects of choice are lotteries:

$$L = (\mu_{\acute{e}}^{L} \colon X \to \{a + b\acute{e} \mid \lfloor a + b\acute{e} \rfloor \leq 1; 0 \leq a, b; \sum_{x \in \mathrm{X}} \left\lfloor \mu_{\acute{e}}^{L}(x) \right\rfloor = 1\})$$

3.1 assumptions

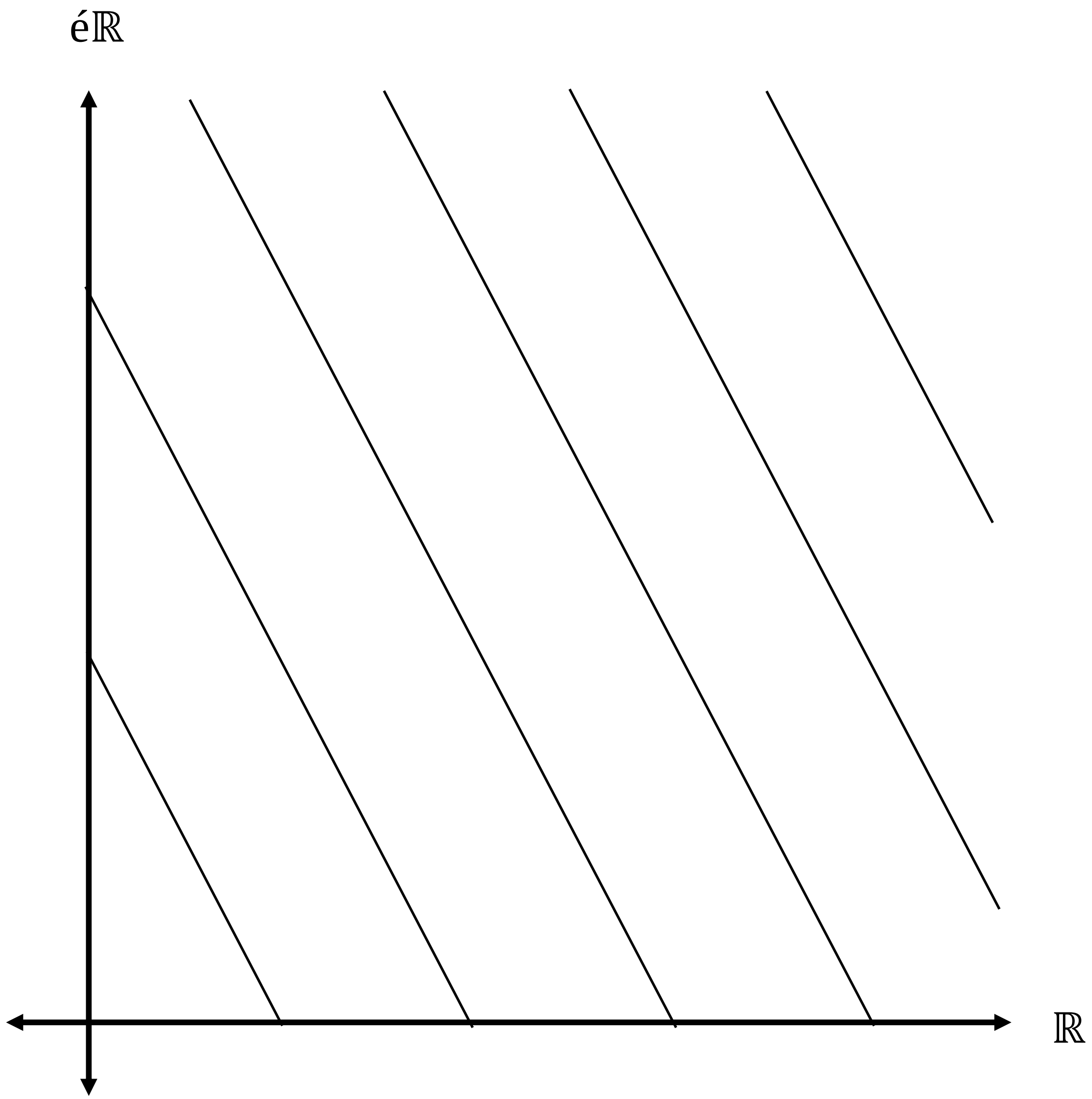
Decision makers satisfy three assumptions; specifically each decision maker:

1. has a real-valued utility function over money: u

2. computes expected utility: $v(L) = \sum_{x \in \mathrm{X}} \mu_{\acute{e}}^{L}(x) \cdot u(x)$

3. converts ambiguous to real utility at a rate: $\alpha > 0$; i.e.: $a_L + b_L \acute{e} \mapsto \alpha a_L + b_L$

Where: $v(L) = a_L + b_L \acute{e}$ and $\phi(v(L)) = \alpha a_L + b_L$



$\alpha > 1$

3.2 the maximand

Decision makers maximize:

$$\max_{L\in\mathcal{L}} \ \phi\left(\sum_{x\in \mathrm{X}} \mu_{\acute{e}}(x)\cdot u(x)\right)$$

Which is equivalent to:

$$\max_{L\in\mathcal{L}}\left(\underbrace{\sum_{x\in \mathrm{X}} \lfloor\mu_{\acute{e}}(x)\rfloor\cdot u(x)}_{\text{EU}} + (\alpha-1)\underbrace{\sum_{x\in \mathrm{X}} \mathrm{Re}(\mu_{\acute{e}}(x))\cdot u(x)}_{\text{‘reliable’ EU}}\right)$$

Ellsberg’s formula
Robust Bayesianism

3.3 special cases

There are two signal cases which result in the maximization problem being reduced to the canonical case of expected utility maximization.

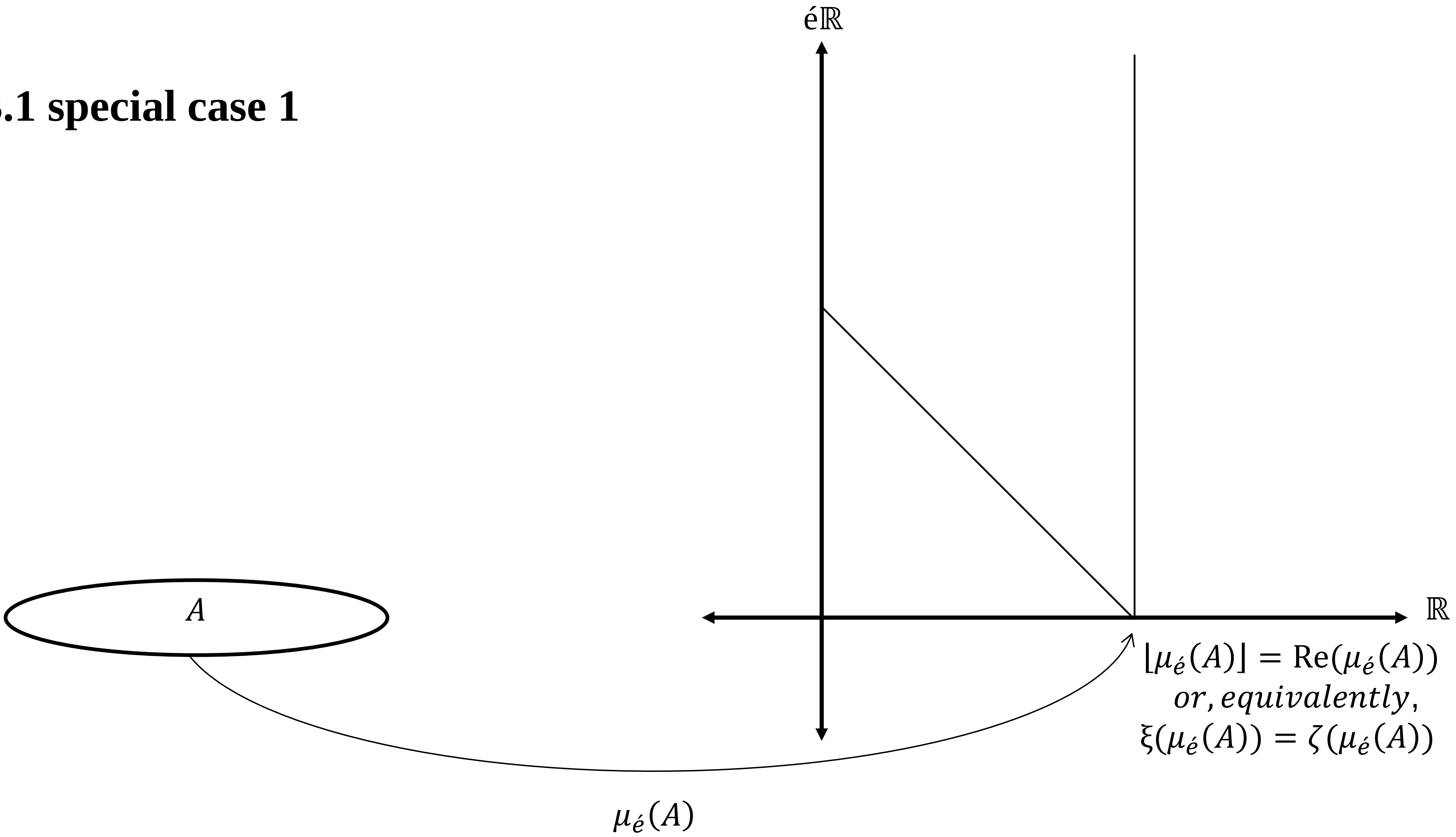
Case 1: all relevant beliefs are reliable: $\lfloor \mu_{\acute{e}}(x) \rfloor = \mathrm{Re}(\mu_{\acute{e}}(x))$ for all x.

('objective probabilist' = objectivist)

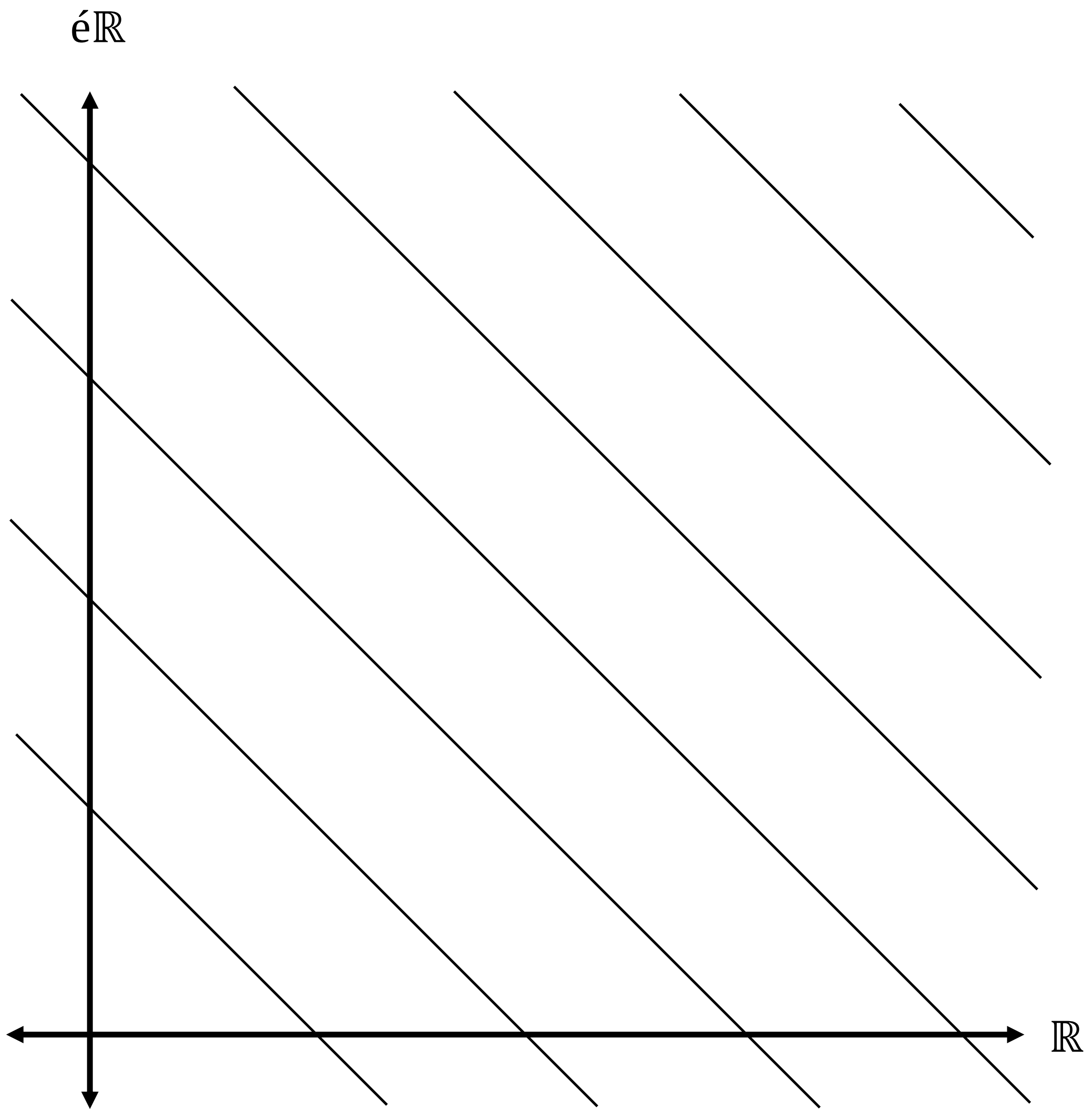
Case 2: the decision maker is ambiguity neutral: $\alpha = 1$.

('subjective probabilist' = subjectivist)

3.3.1 special case 1



3.3.2 special case 2



3.4 special cases

$$\left.\begin{array}{lcl} objectivist & \rightarrow & \xi \circ \mu_{\acute{e}}(A) = \zeta \circ \mu_{\acute{e}}(A) \quad \forall A \\ subjectivist & \rightarrow & \xi \circ v(L) = \psi \circ v(L) \quad \forall L \end{array}\right\} \rightarrow \quad expected\ utility\ maximizer$$

$$pluralist \quad \rightarrow \quad neither \left\{\begin{array}{c} \xi \circ \mu_{\acute{e}}(A) = \zeta \circ \mu_{\acute{e}}(A) \quad \forall A \\ nor \\ \xi \circ v(L) = \psi \circ v(L) \quad \forall L \end{array}\right\} \rightarrow \quad generalized\ expected\ utility\ maximizer$$

3.4.1 special cases

		Ambiguity conservative	Ambiguity liberal
		Ambiguity neutral	**Ambiguity avid/averse**
Measure conservative	**Events measurable**	Rigorist	Objectivist
Measure liberal	**Events not measurable**	Subjectivist	Pluralist

3.5 Ambiguity and gambling

	Heads	Tails
White	(0,¼)	(0,¼)
Black	(0,¼)	(0,¼)

If you guess correctly which tin contains the Benjamin, you win it!

3.5 ambiguity and gambling

Here are two questions:

1. How much would you pay to be told how many marbles of each colour are in the urn before choosing?
2. How much would you pay to have *either* the risk *or* the uncertainty resolved before choosing?

3.5.1 ambiguity and gambling

Grid expected value, pre-offer: $\lfloor .25\acute{e} \times 100 \rfloor = 25$

Grid expected value, post-offer: $\lfloor .5\acute{e} \times .5 \times 100 \rfloor + \lfloor .5\acute{e} \times .25 \times 100 \rfloor = 37.5$

Ambiguity neutral decision maker pays:* $12.50

Ambiguity averse decision maker pays:* $11.36 = (12.5/1.1) ($\alpha = 1.1$)

*We assume risk neutrality.

3.5.2 ambiguity and gambling

Grid expected value, pre-offer: $\lfloor .25\acute{e} \times 100 \rfloor = 25$

Grid expected value, post-offer: $\lfloor .5 \times 100 \rfloor = 50$

Ambiguity neutral decision maker pays:* $\$25.00$

Ambiguity averse decision maker pays:* $\$27.27$ = 50 – (25/1.1) $(\alpha = 1.1)$

*We assume risk neutrality.

3.5.3 an apparent 'paradox'

Case 1: the *ex ante reliability content* of the new gamble is static even as the grid-expected value rises; hence the ambiguity averse decision maker gets less value out of the new gamble than the ambiguity neutral decision maker

Case 2: the *ex ante reliability content* of the new gamble increases along with the increase in grid-expected value; hence the ambiguity averse decision maker gets more value out of the new gamble than the ambiguity neutral decision maker

3.5.3 an apparent 'paradox'

The ambiguity averse individual may pay less *to reduce the amount of ambiguity than the ambiguity neutral individual, but may pay* more *to insure against ambiguity than the ambiguity neutral individual.*

The reason for this is that the simple promise of having more information revealed doesn't necessarily increase the *weight* or the *reliability* of the decision maker's probability judgments, *ex ante*; and this is true even though she thinks that the reliability of her probability judgments will necessarily be increased by the revelation of the information, *ex post*. Conversely, insurance necessarily increases the reliability of expected utility.

3.5.4 climate change

Planet Earth

Twin Earth

3.5.5 car rental

Is this your first time here?

Oh, you’ll love it – but the roads are terrible!

Partition the sample space if possible.