The Unit-effect Normalisation in Set-identified Structural Vector Autoregressions

Matthew Read

Economic Research Department, Reserve Bank of Australia

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Motivation

- Set-identified SVARs typically assume structural shocks have unit standard deviation.
  - Impulse responses are to 'standard deviation' shocks.
- Responses to 'unit shocks' often more relevant.
  - Example: 100 basis point shock to federal funds rate.
  - Obtained under 'unit-effect normalisation' (e.g. Stock and Watson 2016, 2018).
- Straightforward to swap between normalisations when conducting 'standard' Bayesian inference. But:
  - Problems with Bayesian inference in set-identified models.
  - Properties of other approaches unclear when parameter of interest is impulse response to unit shock.

Contribution

- Show that *identified sets* for impulse responses to unit shocks may be unbounded.
    - Set-identifying restrictions may be extremely uninformative about impulse responses to unit shocks.
- Discuss implications for conducting ‘prior-robust’ Bayesian inference (Giacomini and Kitagawa 2021).
- Show how to check for unboundedness in practice.
- Demonstrate empirical relevance by estimating effects of 100 basis point shock to federal funds rate.

SVAR

The 'orthogonal reduced form' of the SVAR is

$$\mathbf{y}_t = \mathbf{B}\mathbf{x}_t + \underbrace{\boldsymbol{\Sigma}_{tr}\mathbf{Q}\boldsymbol{\varepsilon}_t}_{\mathbf{u}_t \sim N(\mathbf{0}_{n\times 1}, \boldsymbol{\Sigma})}, \quad \boldsymbol{\varepsilon}_t \sim N(\mathbf{0}_{n\times 1}, \mathbf{I}_n),$$

where $\boldsymbol{\Sigma}_{tr}\boldsymbol{\Sigma}_{tr}' = \boldsymbol{\Sigma}$ and $\mathbf{Q} \in \mathcal{O}(n)$ (i.e. $\mathbf{Q}\mathbf{Q}' = \mathbf{I}_n$).

Denote reduced-form parameters by $\phi = (\mathrm{vec}(\mathbf{B})', \mathrm{vech}(\boldsymbol{\Sigma}_{tr})')'$.

Impulse Responses

- Impulse responses to standard-deviation shocks are obtained from vector moving average representation.
- Denote horizon-h impulse response of ith variable to jth shock by $\eta_{i,j,h}(\phi, \mathbf{Q})$.
- **Definition (impulse response to unit shock):**

$$\tilde{\eta}_{i,1,h}(\phi, \mathbf{Q}) = \frac{\eta_{i,1,h}(\phi, \mathbf{Q})}{\eta_{1,1,0}(\phi, \mathbf{Q})},$$

i.e. horizon-h impulse response of ith variable to shock that raises first variable by one unit on impact.

Identified Sets

- Represent sign restrictions by $S(\phi, \mathbf{Q}) \geq \mathbf{0}_{s\times 1}$ and zero restrictions by $F(\phi, \mathbf{Q}) = \mathbf{0}_{f\times 1}$.

- Identified set (IS) for $\mathbf{Q}$ collects observationally equivalent parameter values:

$$\mathcal{Q}(\phi|S, F) = \left\{\mathbf{Q} \in \mathcal{O}(n) : S(\phi, \mathbf{Q}) \geq \mathbf{0}_{s\times 1}, F(\phi, \mathbf{Q}) = \mathbf{0}_{f\times 1}\right\}.$$

- IS for impulse responses are

$$\eta_{i,j,h}(\phi|S, F) = \left\{\eta_{i,j,h}(\phi, \mathbf{Q}) : \mathbf{Q} \in \mathcal{Q}(\phi|S, F)\right\}$$
$$\tilde{\eta}_{i,1,h}(\phi|S, F) = \left\{\tilde{\eta}_{i,1,h}(\phi, \mathbf{Q}) : \mathbf{Q} \in \mathcal{Q}(\phi|S, F)\right\}.$$

Unboundedness of Identified Sets

$$\tilde{\eta}_{i,1,h}(\phi, \mathbf{Q}) = \frac{\eta_{i,1,h}(\phi, \mathbf{Q})}{\eta_{1,1,0}(\phi, \mathbf{Q})}.$$

- When $0 \in \eta_{1,1,0}(\phi|S, F)$, $\tilde{\eta}_{i,1,h}(\phi|S, F)$ may be unbounded.
- Intuition: consider sequence for $\mathbf{Q}$ converging to $\eta_{1,1,0}(\phi, \mathbf{Q}) = 0$...
- Implication: set-identifying restrictions may be extremely uninformative about impulse responses to unit shocks!

Implications for Robust Bayesian Inference

- Decompose prior for $\boldsymbol{\theta} = (\phi, \mathbf{Q})$ as $\pi_{\boldsymbol{\theta}} = \pi_{\phi}\pi_{\mathbf{Q}|\phi}$.
- Giacomini and Kitagawa (2021) consider class of all $\pi_{\mathbf{Q}|\phi}$ consistent with restrictions:

$$\Pi_{\mathbf{Q}|\phi} = \left\{\pi_{\mathbf{Q}|\phi} : \pi_{\mathbf{Q}|\phi}(\mathcal{Q}(\phi|S, F)) = 1\right\}.$$

- Combine $\Pi_{\mathbf{Q}|\phi}$ with reduced-form posterior $\pi_{\phi|\mathbf{Y}}$:

$$\Pi_{\boldsymbol{\theta}|\mathbf{Y}} = \left\{\pi_{\boldsymbol{\theta}|\mathbf{Y}} = \pi_{\mathbf{Q}|\phi}\pi_{\phi|\mathbf{Y}} : \pi_{\mathbf{Q}|\phi} \in \Pi_{\mathbf{Q}|\phi}\right\}.$$

- $\Pi_{\boldsymbol{\theta}|\mathbf{Y}}$ induces class of posteriors for $\eta/\tilde{\eta}$, $\Pi_{\eta|\mathbf{Y}}$.
- Summarise $\Pi_{\eta|\mathbf{Y}}$ in various ways.
- Boundedness of these summaries will depend on posterior probability that IS is bounded.

Checking for Unboundedness

- *Necessary* condition for unbounded identified sets: $\tilde{\eta}_{i,1,h}(\phi|S,F)$ is unbounded only if $0 \in \eta_{1,1,0}(\phi|S,F)$.
- *Sufficient* condition for $0 \in \eta_{1,1,0}(\phi|S,F)$: if $s+f \leq n$, then $0 \in \eta_{1,1,0}(\phi|S,F)$.
- When sufficient condition not satisfied, numerically check whether $0 \in \eta_{1,1,0}(\phi|S,F)$.
- Report posterior probability that $0 \in \eta_{1,1,0}(\phi|S,F)$.

Estimating the Effects of US Monetary Policy

- Estimate effects of 100 basis point federal funds rate shock.
- Model and identifying restrictions are from Uhlig (2005) and Arias, Caldara and Rubio-Ramírez (2019).
- $\mathbf{y}_t = (FFR_t, GDP_t, GDPDEF_t, COM_t, TR_t, NBR_t)'$.

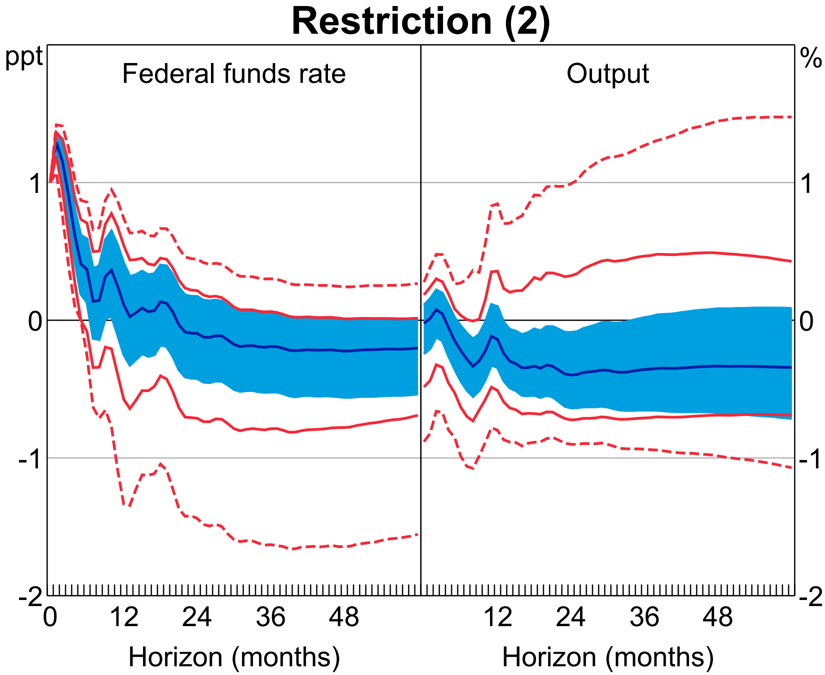
Restriction (1)

- Restrictions on 'systematic component of monetary policy' from Arias, Caldara and Rubio-Ramírez (2019):
  - Reserves do not enter monetary policy reaction function (zero restrictions).
  - FFR_t not decreased in response to higher output or prices (sign restrictions).
  - FFR_t does not decrease in response to positive monetary policy shock (sign restriction) + sign normalisation.

- $s + f = 4 + 2 \leq n = 6$ – IS for impact response of FFR_t always includes zero.

Restriction (2)

- Combine Restriction (1) with sign restrictions on impulse responses from Uhlig (2005).
- In 6 months after monetary policy shock, FFR_t does not decrease, and $GDPDEF_t$, COM_t and NBR_t do not increase.
- $f = 2$ and $s = 27$ – sufficient condition not satisfied.
- IS for impact response of FFR_t includes zero with low posterior probability (< 0.1 per cent).


Restriction (2)
ppt
%
Federal funds rate
Output
1
0
-1
-2
0
12
24
36
48
Horizon (months)
Horizon (months)

Conclusion

- Set-identifying restrictions may be extremely uninformative about impulse responses to unit shocks, particularly when few restrictions.
- This is relevant empirically.
- Thanks for listening!

Spares

Impulse Responses

- Impulse responses to standard-deviation shocks are obtained from coefficients of VMA representation:

$$y_t = \sum_{h=0}^{\infty} \mathbf{C}_h \mathbf{\Sigma}_{tr} \mathbf{Q} \varepsilon_{t-h},$$

  where $\{\mathbf{C}_h\}_{h=0}^{\infty}$ depends on $\mathbf{B}$.
- Horizon-h impulse response of ith variable to jth shock is

$$\eta_{i,j,h}(\phi, \mathbf{Q}) = \mathbf{e}'_{i,n} \mathbf{C}_h \mathbf{\Sigma}_{tr} \mathbf{Q} \mathbf{e}_{j,n}.$$

- Horizon-h impulse response of ith variable to shock that raises first variable by one unit on impact is

$$\tilde{\eta}_{i,1,h}(\phi, \mathbf{Q}) = \frac{\eta_{i,1,h}(\phi, \mathbf{Q})}{\eta_{1,1,0}(\phi, \mathbf{Q})}.$$

Bivariate Model

- Model is $\mathbf{y}_t = \boldsymbol{\Sigma}_{tr}\mathbf{Q}\varepsilon_t$ with $\mathbb{E}(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t') = \mathbf{I}_2$.
- Let $\boldsymbol{\phi} = \mathrm{vech}(\boldsymbol{\Sigma}_{tr}) = (\sigma_{11}, \sigma_{21}, \sigma_{22})'$ with $\sigma_{11}, \sigma_{22} > 0$.
- Space of 2×2 orthonormal matrices can be represented as

$$\mathcal{O}(2) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \right\},$$

  where $\theta \in [-\pi, \pi]$.
- Consider sign restrictions $\eta_{1,1,0} \geq 0$ and $\eta_{2,1,0} \leq 0$.

Impulse Responses in Bivariate Example

IS for $\mathbf{A}_0^{-1}$ is

$$\mathbf{A}_0^{-1} \in \left\{ \begin{bmatrix} \sigma_{11}\cos\theta & -\sigma_{11}\sin\theta \\ \sigma_{21}\cos\theta + \sigma_{22}\sin\theta & \sigma_{22}\cos\theta - \sigma_{21}\sin\theta \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} \sigma_{11}\cos\theta & \sigma_{11}\sin\theta \\ \sigma_{21}\cos\theta + \sigma_{22}\sin\theta & \sigma_{21}\sin\theta - \sigma_{22}\cos\theta \end{bmatrix} \right\}.$$

Response of second variable to 'unit shock' in first variable is

$$\tilde{\eta}_{2,1,0} \equiv \frac{\eta_{2,1,0}}{\eta_{1,1,0}} = \frac{\sigma_{21}\cos\theta + \sigma_{22}\sin\theta}{\sigma_{11}\cos\theta} = \frac{\sigma_{21}}{\sigma_{11}} + \frac{\sigma_{22}}{\sigma_{11}}\tan\theta.$$

Identified Sets in Bivariate Model

$$\eta_{1,1,0} \in \begin{cases} \left[\sigma_{11} \cos\left(\arctan\left(\min\left\{\frac{\sigma_{22}}{\sigma_{21}}, \frac{\sigma_{21}}{\sigma_{22}}\right\}\right)\right), \sigma_{11}\right] & \text{if } \sigma_{21} < 0 \\ \left[0, \sigma_{11} \cos\left(\arctan\left(-\frac{\sigma_{21}}{\sigma_{22}}\right)\right)\right] & \text{if } \sigma_{21} \geq 0. \end{cases}$$

$$\tilde{\eta}_{2,1,0} \in \begin{cases} \left[\frac{\sigma_{21}}{\sigma_{11}} + \frac{\sigma_{22}^2}{\sigma_{11}\sigma_{21}}, 0\right] & \text{if } \sigma_{21} < 0 \\ (-\infty, 0] & \text{if } \sigma_{21} \geq 0. \end{cases}$$

- $\tilde{\eta}_{2,1,0}(\phi|S)$ is unbounded when $0 \in \eta_{1,1,0}(\phi|S)$.
- Set-identifying restrictions may be very uninformative about impulse responses to unit shocks!

Implications for Robust Bayesian Inference

- Boundedness of inferential outputs depends on posterior probability that IS is bounded (α). For example:
  - Set of posterior *means* is unbounded unless $\alpha = 1$.
  - Set of posterior *medians* is unbounded unless $\alpha \geq 0.5$.
- Can still draw useful posterior inferences when IS unbounded with positive probability.
- Knowing α tells us which inferential outputs are bounded, and thus about the informativeness of the restrictions.

Implications for Robust Bayesian Inference

- Assume $\pi_{\phi|\mathbf{Y}}$ is supported on two values of ϕ:

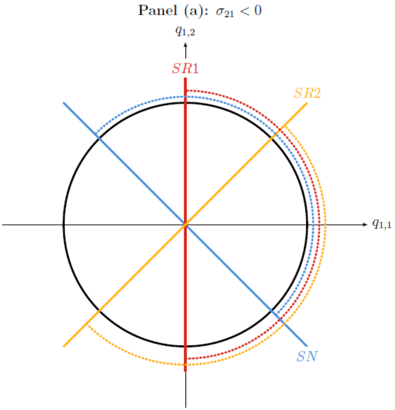
$$\phi^a = (\sigma_{11}, \sigma_{21}^a, \sigma_{22})' \quad \text{with } \sigma_{21}^a < 0$$
$$\phi^b = (\sigma_{11}, \sigma_{21}^b, \sigma_{22})' \quad \text{with } \sigma_{21}^b \geq 0.$$

- Let $\alpha = \pi_{\phi|\mathbf{Y}}(\phi = \phi^a)$ and $\ell(\phi^a) = \inf \tilde{\eta}_{2,1,0}(\phi^a|S)$.
- Set of posterior *means* is $(-\infty, 0]$ unless $\alpha = 1$.
- Set of posterior *medians* is $[\ell(\phi^{(a)}), 0]$ for $\alpha \geq 0.5$ and $(-\infty, 0]$ otherwise.
- *Posterior lower/upper probabilities* always well-defined.
- Takeaway: can still draw useful posterior inferences when IS (sometimes) unbounded.

Geometric Illustration in Bivariate Model

No dynamics, $\boldsymbol{\phi} = \mathrm{vech}(\boldsymbol{\Sigma}_{tr}) = (\sigma_{11}, \sigma_{21}, \sigma_{22})'$.

$\eta_{1,1,0} \geq 0$ (SR1), $\eta_{2,1,0} \leq 0$ (SR2) + normalisation (SN).



Necessary Condition for Unboundedness

Proposition 4.1. Assume $\mathcal{Q}(\phi|S, F)$ is nonempty and interest is in the impulse response to a unit shock in the first variable at some fixed and finite horizon h. The identified set for the impulse response to a unit shock to the first variable, $\tilde{\eta}_{i,1,h}(\phi|S, F)$, is unbounded only if $0 \in \eta_{1,1,0}(\phi|S, F)$.

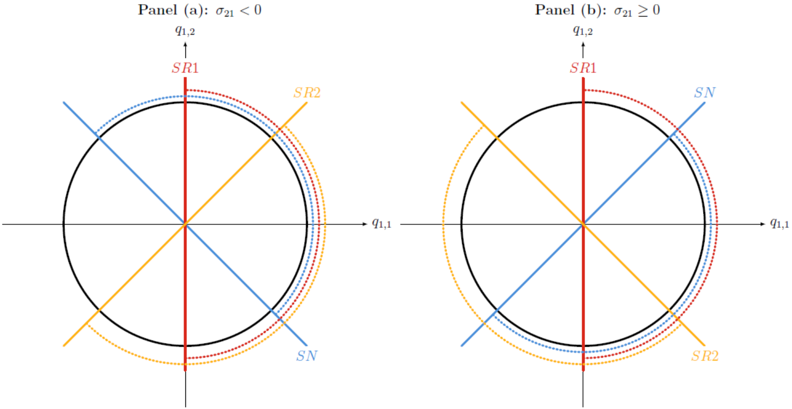
A Useful Sufficient Condition

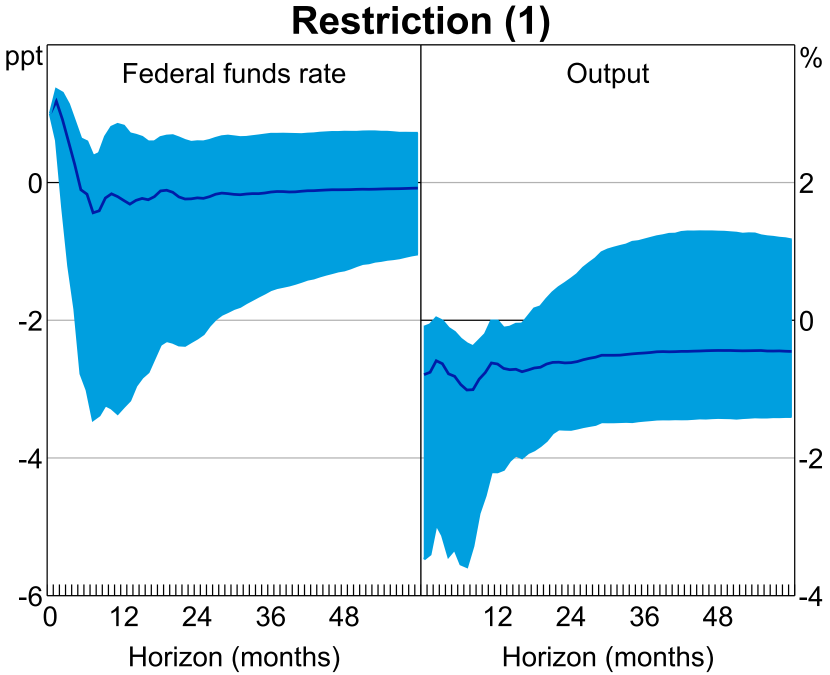
Proposition 4.2. Assume that any sign and zero restrictions constrain $\mathbf{q}_1$ only, $\eta_{1,1,0} = \mathbf{e}'_{1,n}\mathbf{\Sigma}_{tr}\mathbf{q}_1 \geq 0$ is contained within the set of sign restrictions $S(\phi)\mathbf{q}_1 \geq \mathbf{0}_{s\times 1}$ and the number of zero restrictions in $\mathbf{F}(\phi)\mathbf{q}_1 = \mathbf{0}_{f\times 1}$ satisfies $0 \leq f < n-1$. Also, let $\tilde{F}(\phi) = \left(\left(\mathbf{e}'_{1,n}\mathbf{\Sigma}_{tr}\right)', F(\phi)'\right)'$ and assume $\mathrm{rank}(\tilde{F}(\phi)) = f+1$ holds ϕ-almost surely. If $s+f \leq n$, then $0 \in \eta_{1,1,0}(\phi|S,F)$ holds ϕ-almost surely.

Geometric Illustration of Sufficient Condition

$s = 3 > n = 2$, so sufficient condition not satisfied.

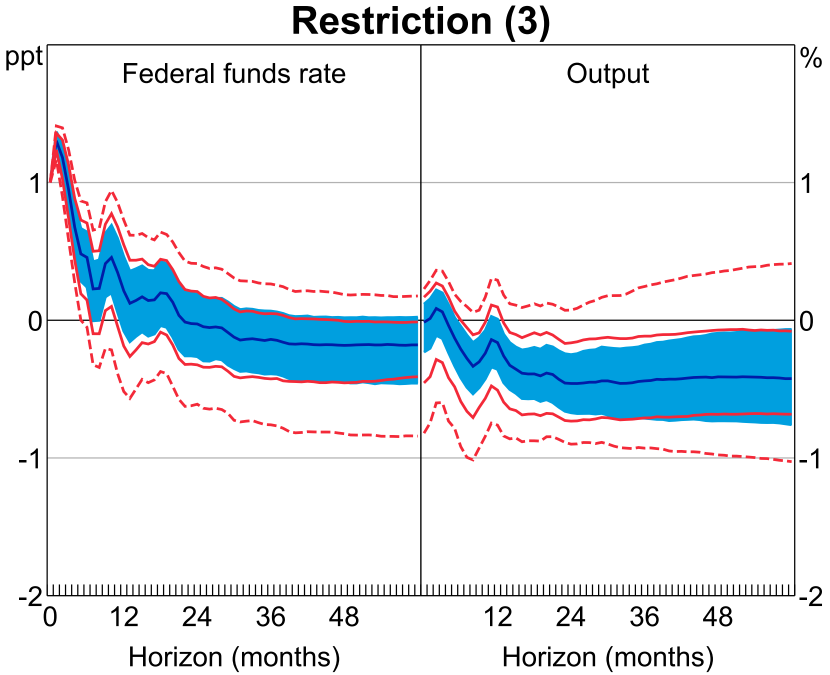
If we delete $SR2$, $s = 2 = n$ and sufficient condition satisfied.




Restriction (1)
ppt
%
Federal funds rate
Output
0
-2
-4
-6
2
0
-2
-4
0
12
24
36
48
12
24
36
48
Horizon (months)
Horizon (months)

Restriction (3)

- Combine Restriction (2) with ‘narrative restrictions’ from Antolín-Díaz and Rubio-Ramírez (2018).
- Monetary policy shock in October 1979 was positive and ‘overwhelming’ contributor to forecast error in FFR_t.
- IS for impact response of FFR_t does not include zero at any draw from reduced-form posterior.



Restriction (3)
ppt
Federal funds rate
Output
%
1
0
-1
-2
0
12
24
36
48
Horizon (months)

Comparison with Estimates from Literature

- Point estimates for largest output response from Ramey (2016) range from 0.6 per cent to 5 per cent.
- Under Restrictions (2) and (3), posterior *upper* probability that output falls by more than 1 per cent after two years is around 5 per cent.
- Consistent with output effects of US monetary policy lying towards the smaller end of the range of existing estimates.

Ruling Out Unboundedness Using Alternative Restrictions

- Imposing identifying restrictions can indirectly rule out unbounded IS.
- Can we rule this out more directly?
  - Magnitude restriction: $\eta_{1,1,0} \geq \lambda$?
  - Lower bound on FEVD?
- Problems:
  - Arguably difficult to elicit lower bound.
  - Estimates may be highly sensitive to imposed bound.

Magnitude Bound in Bivariate Model

- Consider restriction $\eta_{1,1,0} \geq \lambda$ for $0 < \lambda \leq \sigma_{11}$.
- When $\sigma_{21} \geq 0$, lower bound of $\tilde{\eta}_{2,1,0}(\phi|S)$ is

$$\ell(\phi, \lambda) = \frac{\sigma_{21}}{\sigma_{11}} - \frac{\sigma_{22}}{\lambda}\sqrt{\left(1 - \left(\frac{\lambda}{\sigma_{11}}\right)^2\right)}.$$

- Derivative with respect to λ,

$$\frac{\partial \ell(\phi, \lambda)}{\partial \lambda} = \frac{\left(1 - \left(\frac{\lambda}{\sigma_{11}}\right)^2\right)^{-\frac{1}{2}}}{\lambda^2},$$

  tends to ∞ as λ approaches zero from above.
- Lower bound is extremely sensitive to small changes in λ when λ is small.