Two paradoxes of updated beliefs under uncertainty

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1. Aims of the paper

1. Show that the appropriate updating rule for beliefs on $\mathbb{R}\times\mathbb{R}$ is given by the *full Bayesian rule*.

2. Show that the full Bayesian rule operating on $\mathbb{R}\times\mathbb{R}$ yields intuitive results in the situation described by *Gelman's paradox*.

3. Show that there is *another paradox*: ambiguity averse decision makers might pay less to reduce uncertainty than ambiguity neutral decision makers who are otherwise identical.

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2. Algebra

real part

$a+be$ (with: $a,b\in\mathbb{R}$)

imaginary part

$$(a+be)+(c+de)=(a+c)+(b+d)e$$

$$(a+be)\cdot(c+de)=(ac)+(ad+bc+bd)e$$

$$E\cong\mathbb{R}\times\mathbb{R}$$

Algebra – idempotent representation

real part

$a+(a+b)e$ (with: $a,b\in\mathbb{R}$)

imaginary part

$$[a+(a+b)e]+[c+(c+d)e]=(a+c)+(a+b+c+d)e$$

$$[a+(a+b)e]\cdot[c+(c+d)e] = (ac)+(ac+ad+bc+bd)e$$

$$E\cong\mathbb{R}\times\mathbb{R}$$

3. Priors

Definition 3: an *objective information* mapping, $\mathfrak{B}:\mathcal{F}\rightarrow[0,1]$, is a superadditive capacity, normalized to unity ($\mathfrak{B}(\Omega)=1$), whose empty set is null ($\mathfrak{B}(\emptyset)=0$), and which has a non-empty core, denoted: $core(\mathfrak{B})$, which is closed.

Since the core of $\mathfrak{B}$ is non-empty, there exists a set of numbers, $\{\beta(A)\}\downarrow A\in\mathcal{F}$, with $\beta(A)\in[0,1]$, so that $\mathfrak{B}+\beta$ is a canonical probability.

Objective information:

1. $0 \leq \mathfrak{B}(A) \quad \forall A \in \mathcal{F}$

2. $\mathfrak{B}(\Omega) = 1$

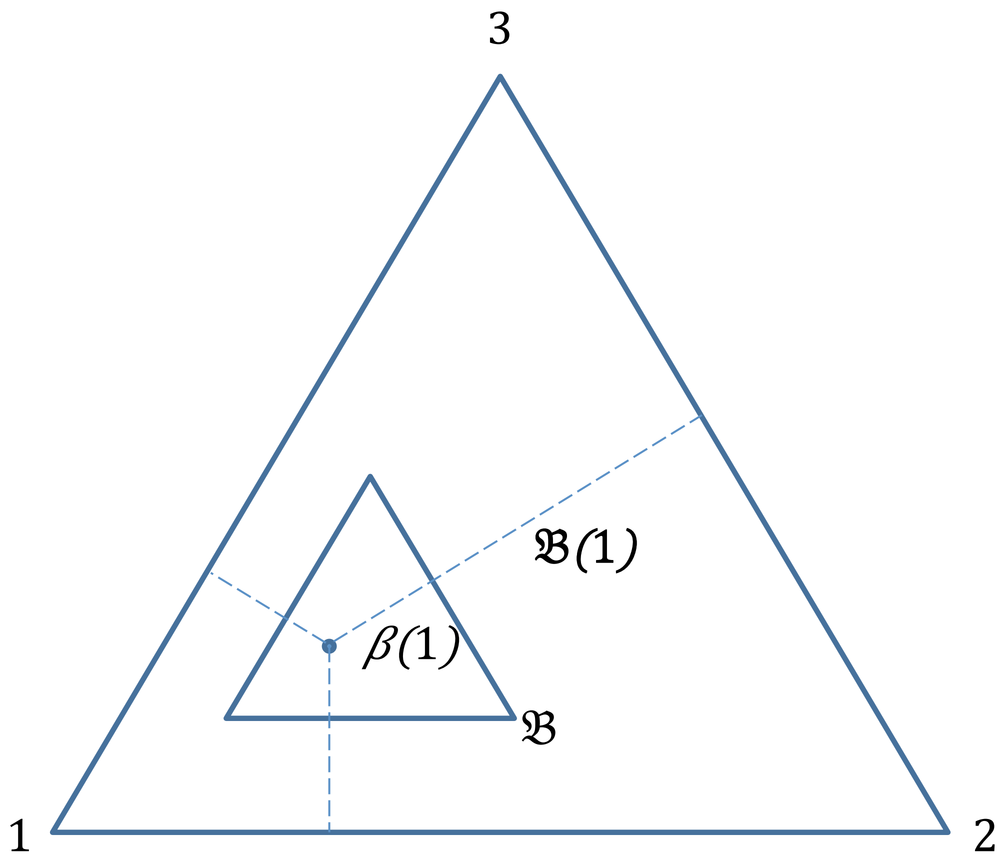
3a. $\mathfrak{B}(A \cup B) \geq \mathfrak{B}(A) + \mathfrak{B}(B)$ *when* $A \cap B = \emptyset$

3b. $\mathfrak{B}(A \cup B) \geq \mathfrak{B}(A) + \mathfrak{B}(B) - \mathfrak{B}(A \cap B)$ (*if also supermodular*)

4. $A \supseteq B \Rightarrow \mathfrak{B}(A) \geq \mathfrak{B}(B)$.

5. $\mathfrak{B}(\emptyset) = 0$.

Simplex of beliefs

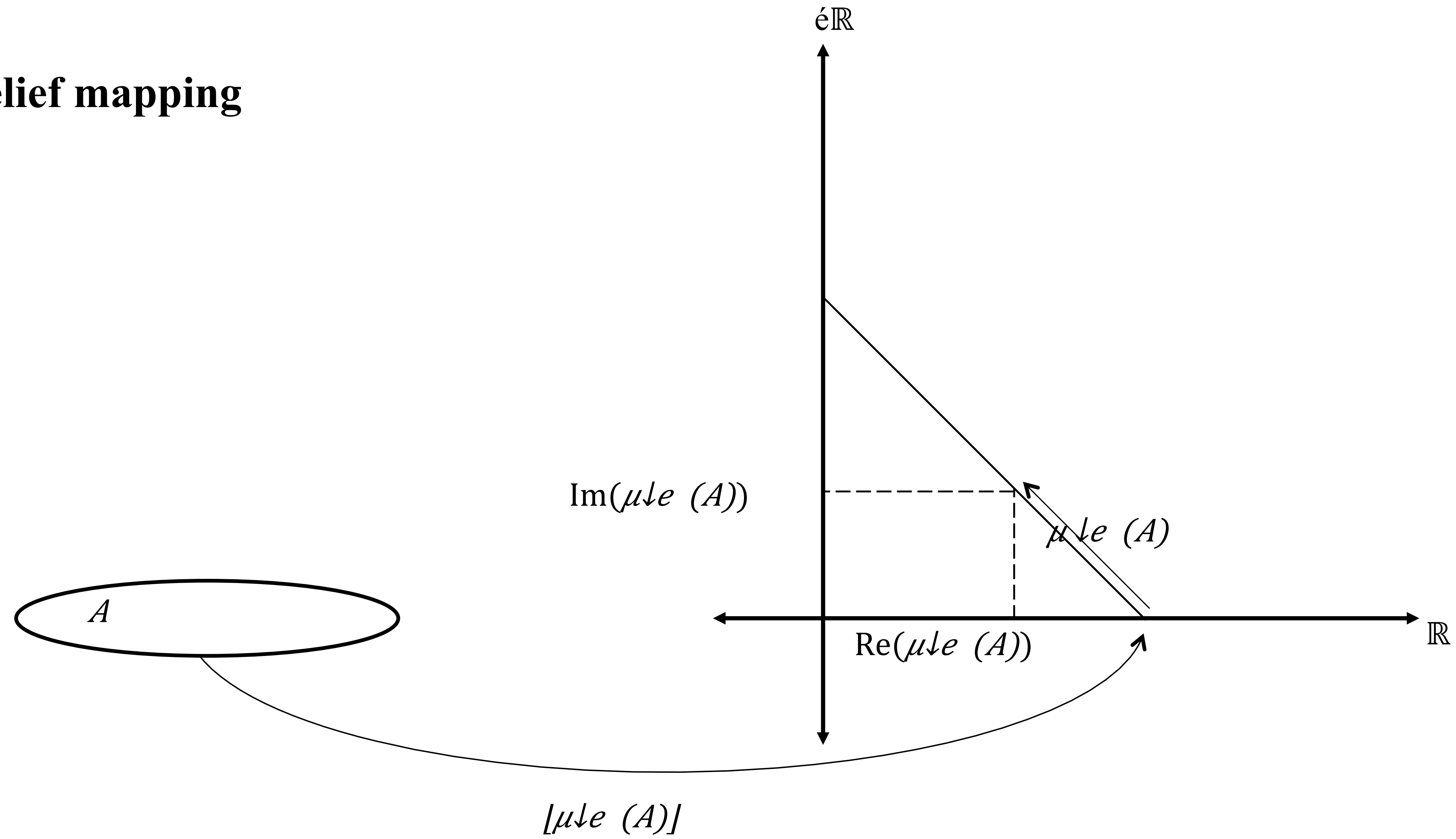


Priors

Definition 4: an *imprecise belief* is a mapping, $\mu_e : \mathcal{F} \to E$, with: $\mu_e(A) = a(A) + b(A)e$, and: $a(A) = \mathfrak{B}(A)$ and: $b(A) = \beta(A)$ for all $A \in \mathcal{F}$.

Note that: $\mathfrak{B}(A) + \beta(A) = \lfloor \mu_e(A) \rfloor$ is a probability.

Belief mapping



Example: the Ellsberg 3 colour problem

$$\mu_e(b) = 0 + 1/3\, e = \mu_e(y),$$

$$\mu_e(r) = 1/3 + 0e$$

$$\mu_e(r \cup b) = 1/3 + 1/3\, e = \mu_e(r \cup y)$$

$$\mu_e(b \cup y) = 2/3 + 0e$$

Example: the boxer, the wrestler and the coin flip*

	Heads	Tails
White	(0,¼)	(0,¼)
Black	(0,¼)	(0,¼)

***Andrew Gelman**

Example: the boxer, the wrestler and the coin flip

$$\mu_{e}(hb) = \mu_{e}(hw) = \mu_{e}(tb) = \mu_{e}(tw) = 0 + 1/4\, e$$

$$\mu_{e}(black) = \mu_{e}(white) = 0 + 1/2\, e$$

$$\mu_{e}(heads) = \mu_{e}(tails) = 1/2 + 0e$$

$$\mu_{e}(hb^{c}) = \mu_{e}(hw^{c}) = \mu_{e}(tb^{c}) = \mu_{e}(tw^{c}) = 1/2 + 1/4\, e$$

Proposition 2:

1. $0 \leq \mu_e(A) \quad \forall A \in \mathcal{F}$

2. $\mu_e(\Omega) = 1$

3a. $[\mu_e(A \cup B)] = [\mu_e(A)] + [\mu_e(B)]$ *when* $A \cap B = \emptyset$

3b. $[\mu_e(A \cup B)] = [\mu_e(A)] + [\mu_e(B)] - [\mu_e(A \cap B)]$

(*if also supermodular*)

4. $A \supseteq B \Rightarrow [\mu_e(A)] \geq [\mu_e(B)]$

Recall that: $[\mu_e] = \mathfrak{B} + \beta$.

5. $\mu_e(\emptyset) = 0$.

Proposition 3:

1. $0 \leq \mu \downarrow e \ (A) \quad \forall A \in \mathcal{F}$
2. $\mu \downarrow e \ (\Omega) = 1$

3a. $\mu \downarrow e \ (A \cup B) \geq \mu \downarrow e \ (A) + \mu \downarrow e \ (B)$ *when* $A \cap B = \emptyset$

3b. $\mu \downarrow e \ (A \cup B) \geq \mu \downarrow e \ (A) + \mu \downarrow e \ (B) - \mu \downarrow e \ (A \cap B)$ (*if also supermodular*)

4. $A \supseteq B \Rightarrow \mu \downarrow e \ (A) \geq \mu \downarrow e \ (B)$
5. $\mu \downarrow e \ (\emptyset) = 0.$

Note that: $\mu \downarrow e = (\mathfrak{B}, [\mu \downarrow e \ ]) = (\mathfrak{B}, \mathfrak{B} + \beta)$.

4. Updating

Definition: $\mathfrak{P}(A)=1-\mathfrak{B}(A\uparrow c\ )$

Definition: $\mu\uparrow e\ \ (A)=\ (1-\mathfrak{B}(A\uparrow c\ ),\ [\mu\downarrow e\ \ (A)])$

$$=1-\mu\downarrow e\ \ (A\uparrow c\ ).$$

Definition: $\mu\downarrow e\ \ AB=(\mathfrak{B}(A|B),[\mu\downarrow e\ \ (A|B)])$

Definition:

$$\mathfrak{B}AB=\mathfrak{B}(A\cap B)/\mathfrak{B}(A\cap B)+\mathfrak{P}(A\uparrow c\cap B)$$

Updating

	B	B^c
A	$A \cap B$	$A \cap B^c$
A^c	$A^c \cap B$	$A^c \cap B^c$

Full Bayesian Updating Rule

Proposition 5:

$$\mu\downarrow e \ \ AB = \mu\downarrow e \ \ (A\cap B)/\mu\downarrow e \ \ (A\cap B)+\mu\uparrow e \ \ (A\uparrow c \cap B)$$

Proposition 6:

1. $0 \le \mu\downarrow e\ \ AB$

2. $\mu\downarrow e\ \ AA = 1$

3a. $\mu\downarrow e\ \ A \cup CB \ge \mu\downarrow e\ \ AB + \mu\downarrow e\ \ CB$ when $A \cap C \ne \emptyset$

3b. $\mu\downarrow e\ \ A \cup CB \ge \mu\downarrow e\ \ AB + \mu\downarrow e\ \ CB - \mu\downarrow e\ \ A \cap CB$

4. $A \supseteq C \Rightarrow \mu\downarrow e\ \ AB \ge \mu\downarrow e\ \ CB$

5. $\mu\downarrow e\ \ \emptyset B = 0.$

5. Independence

Definition: *Scalar independence* of A on B implies: $\mu_e \; AB = \mu_e \; A\Omega \equiv \mu_e \; (A)$.

Note that scalar independence of A on B does not imply scalar independence of B on A.

Definition:

$$\kappa_{A|B} = [\mu_e \; (A \cap B) + \mu^e \; (A^c \cap B) / \mu_e \; (B) \,]$$

Independence

Proposition 9: if A is scalar independent of B, and vice versa, then:

$$\mu_e(A).\,\mu_e(B).\kappa_{A|B} = \mu_e(A).\,\mu_e(B).\kappa_{B|A} = \mu_e(A \cap B).$$

Note that mutual scalar independence implies that: $\kappa_{A|B} = \kappa_{B|A}$.

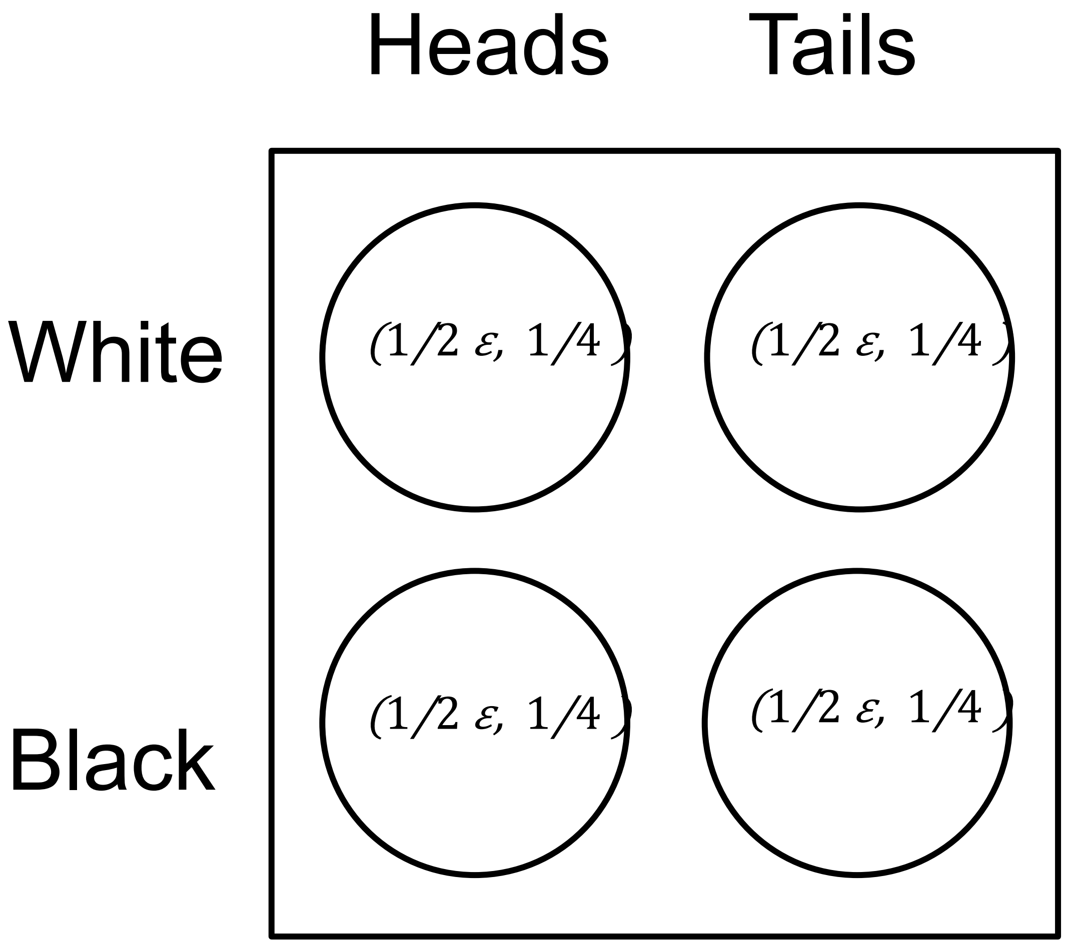
6. Probability chains

Proposition 10:

$$\mu\downarrow e \;\; AB\cap C \,.\, \kappa\downarrow A|BC \,.\, \mu\downarrow e \;\; BC \,.\, \kappa\downarrow B|C = \mu\downarrow e \;\; A\cap BC \,.$$
$$\kappa\downarrow AB|C$$

Remark: the ‘sequential’ updating of imprecise beliefs resembles the updating process for canonical probabilities, but there are also adjustment factors which track the degrees of ambiguity of the relevant conditional beliefs.

7. Paradox no.1: the boxer, the wrestler and the coin flip



Example: the boxer, the wrestler and the coin flip

$$\mu_e(hb) = \mu_e(hw) = \mu_e(tb) = \mu_e(tw) = (\varepsilon/2, \ 1/4)$$

$$\mu_e(black) = \mu_e(white) = (\varepsilon, \ 1/2)$$

$$\mu_e(heads) = \mu_e(tails) = (1/2, \ 1/2)$$

$$\mu_e(hb^c) = \mu_e(hw^c) = \mu_e(tb^c) = \mu_e(tw^c) = (1/2, \ 3/4)$$

Note that heads and black are independent events, and that: $\kappa_{h|b} = \kappa_{b|h} = 1$.

Example: the boxer, the wrestler and the coin flip

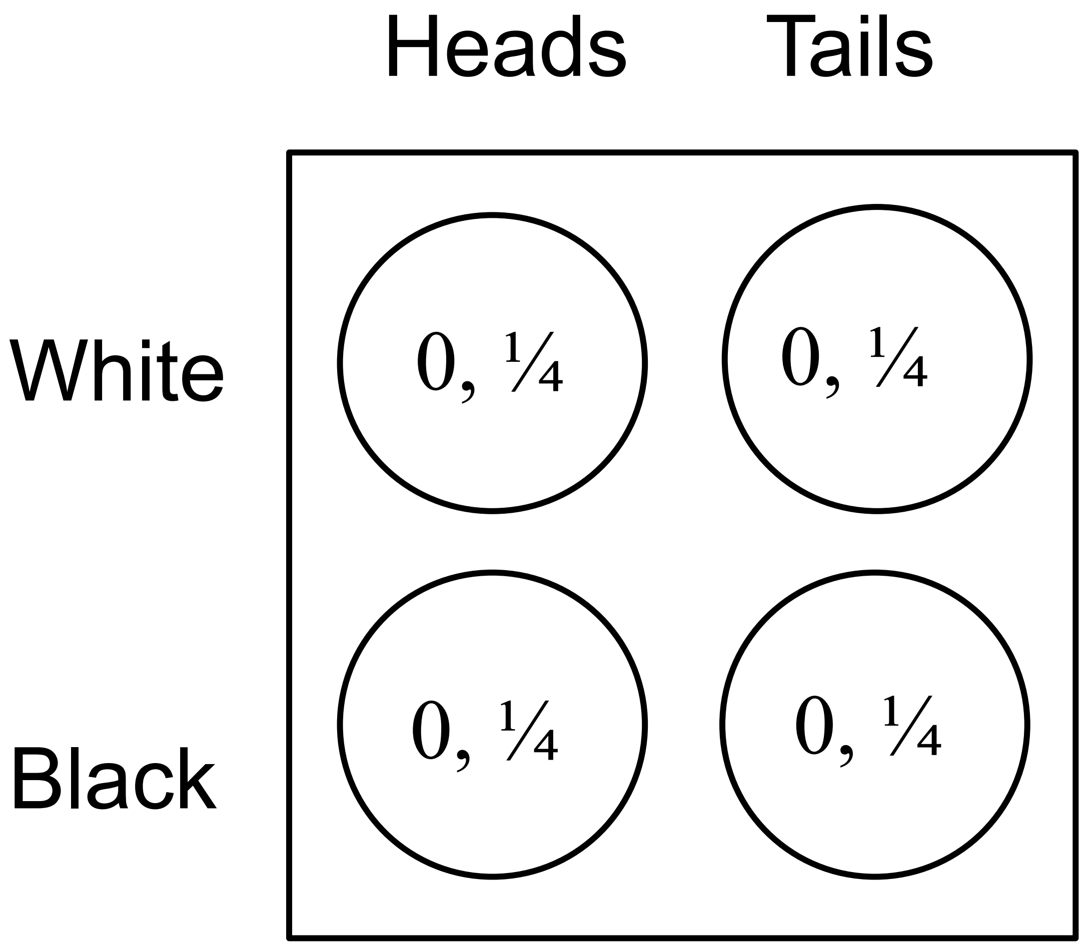
Full Bayesian updating yields:

$$\mu_e(hb|black)=(1/2, 1/2)$$

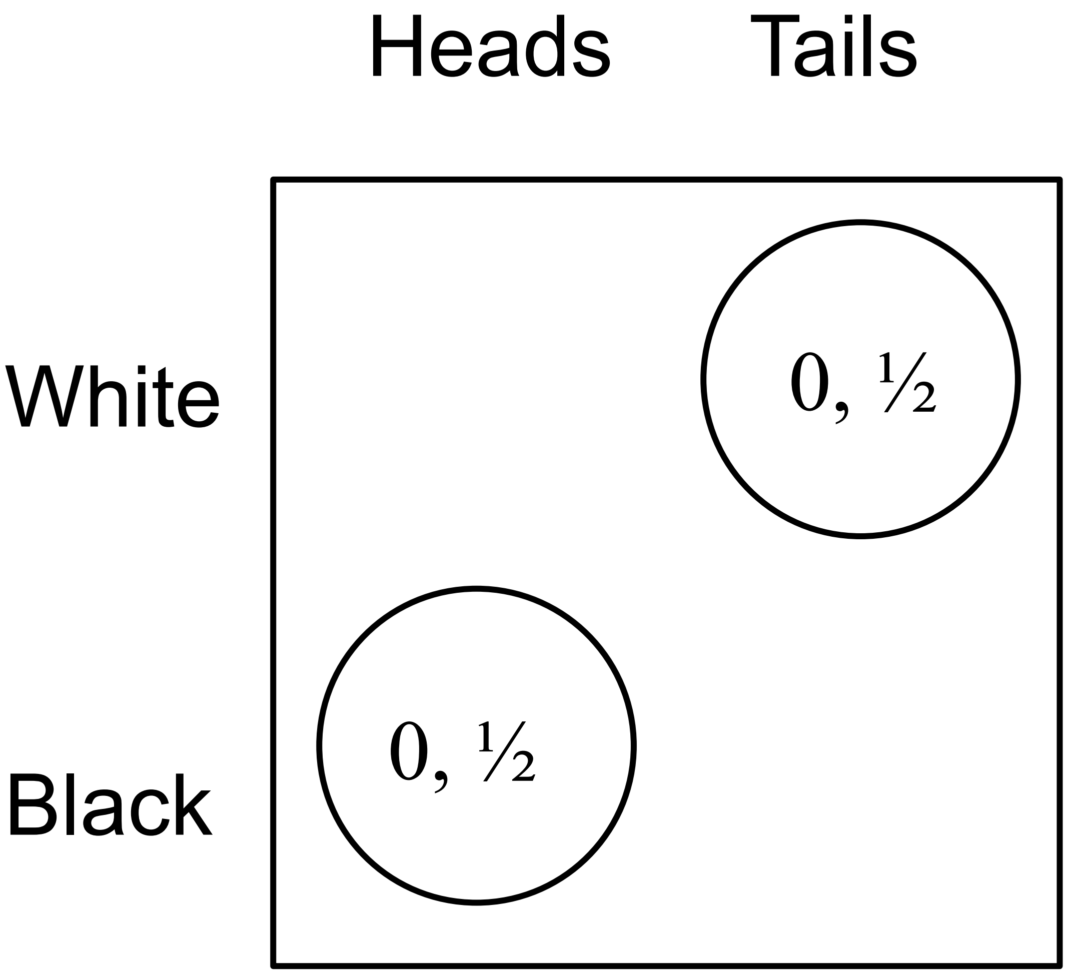
$$\mu_e(hb|heads)=(\varepsilon, 1/2)\approx(0, 1/2)$$

$$\mu_e(hb|hb\cup tw)=(\varepsilon, 1/2)\approx(0, 1/2)$$

Gelman Paradox: full Bayesian rule

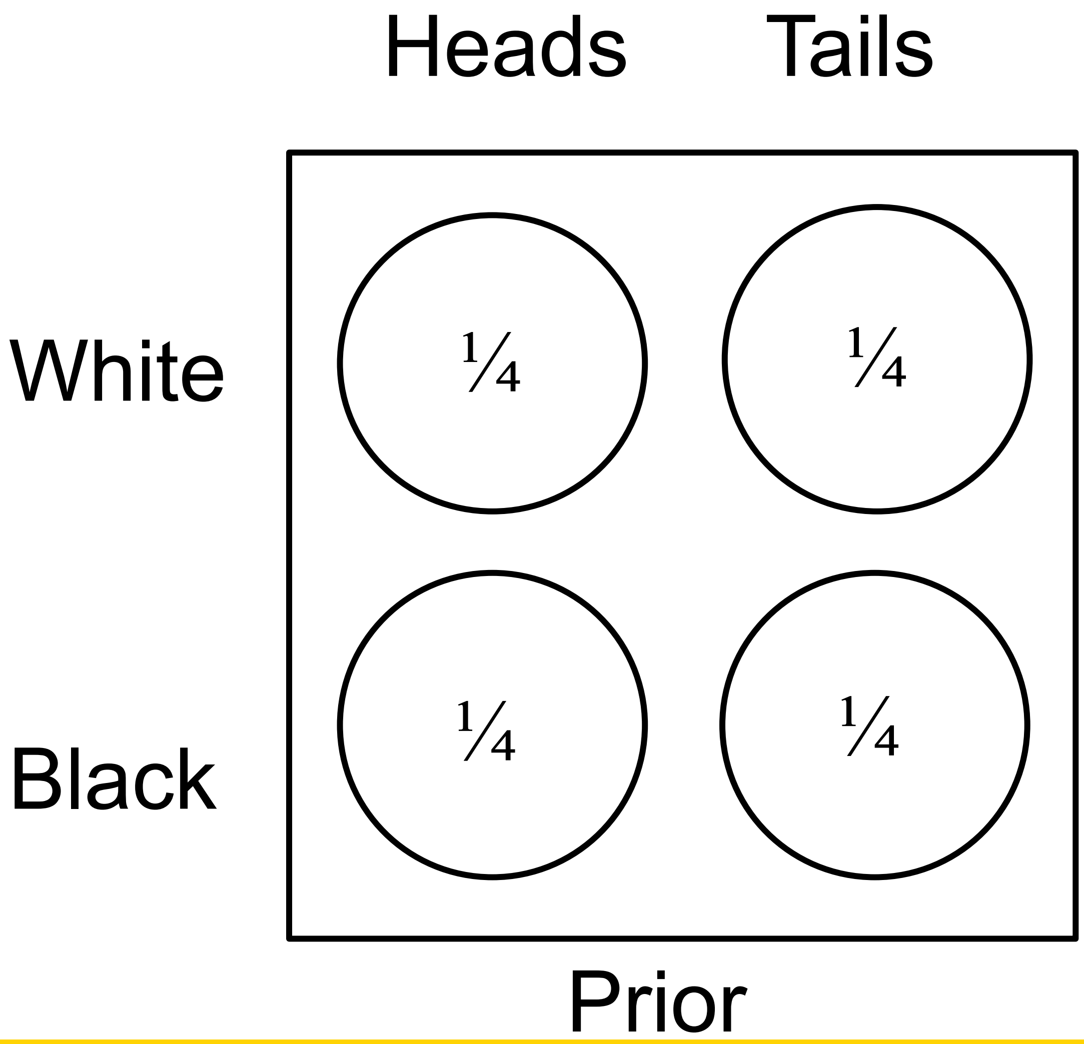


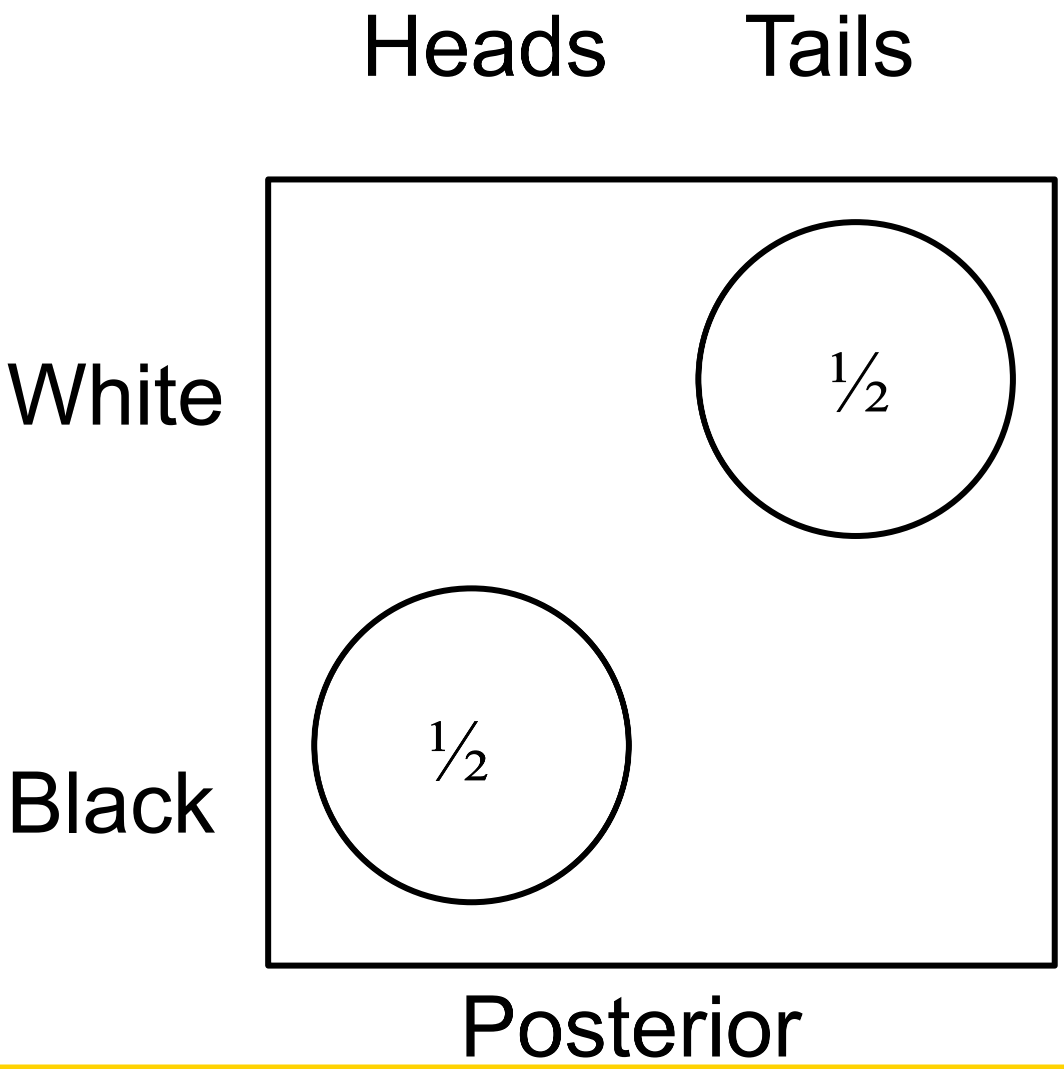
Prior



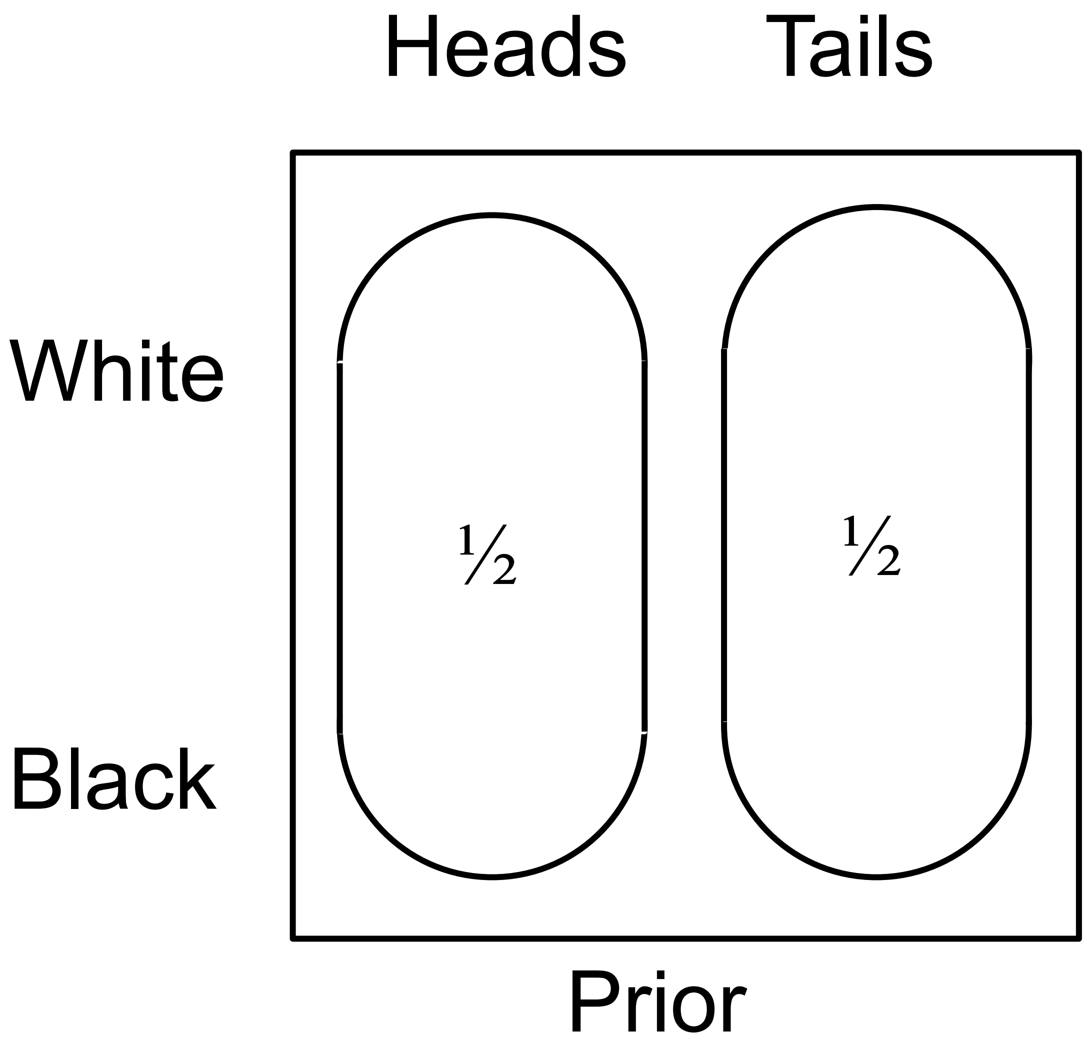
Posterior

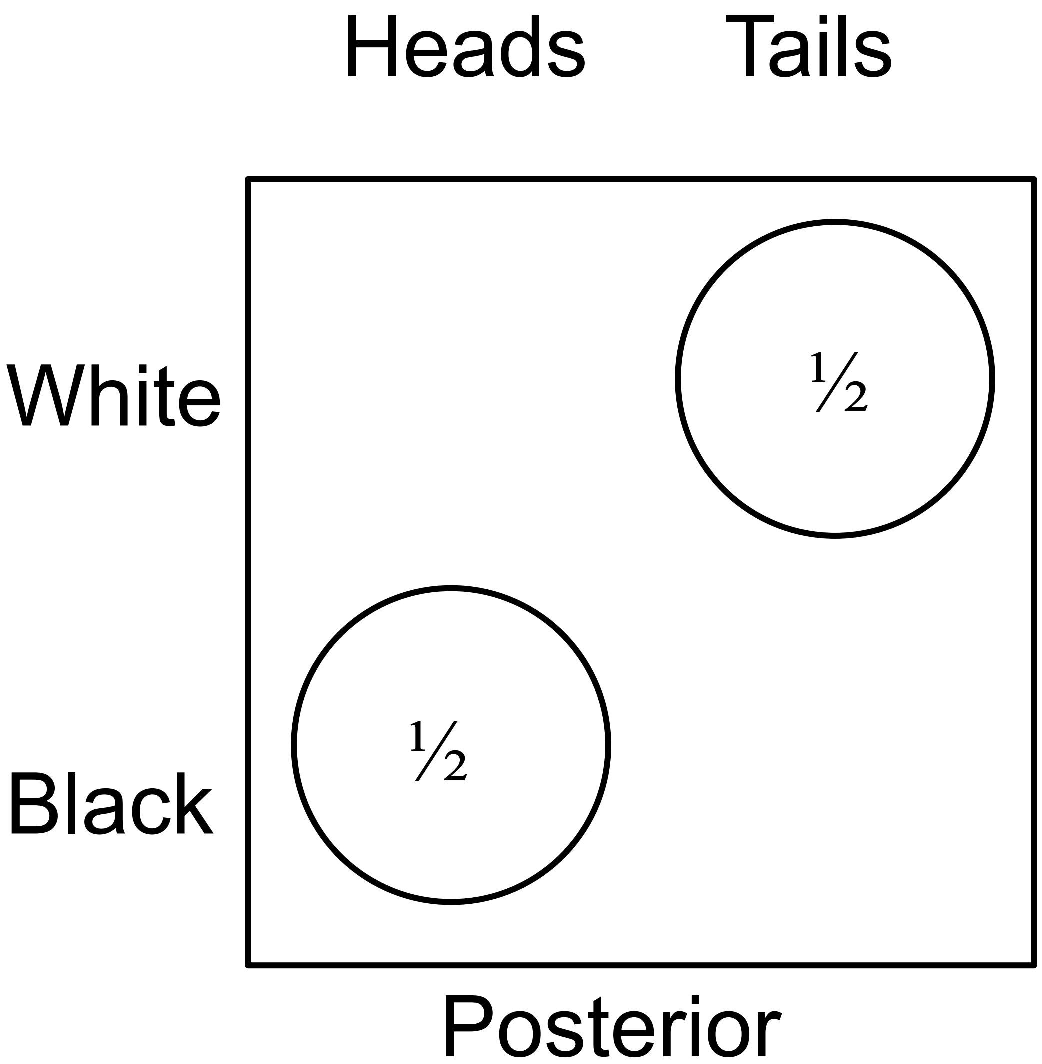
Gelman Paradox: Bayesian rule





Gelman Paradox: Dempster-Shafer rule



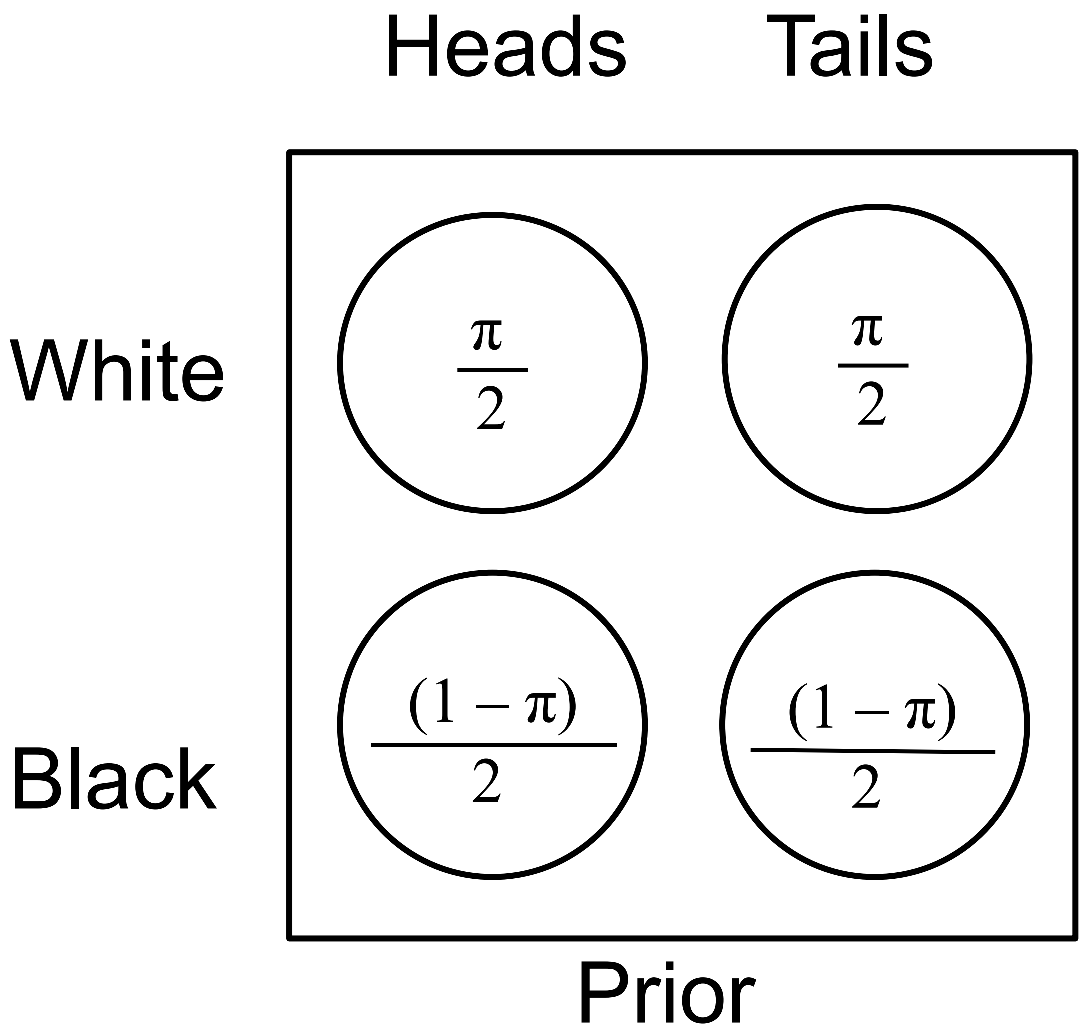


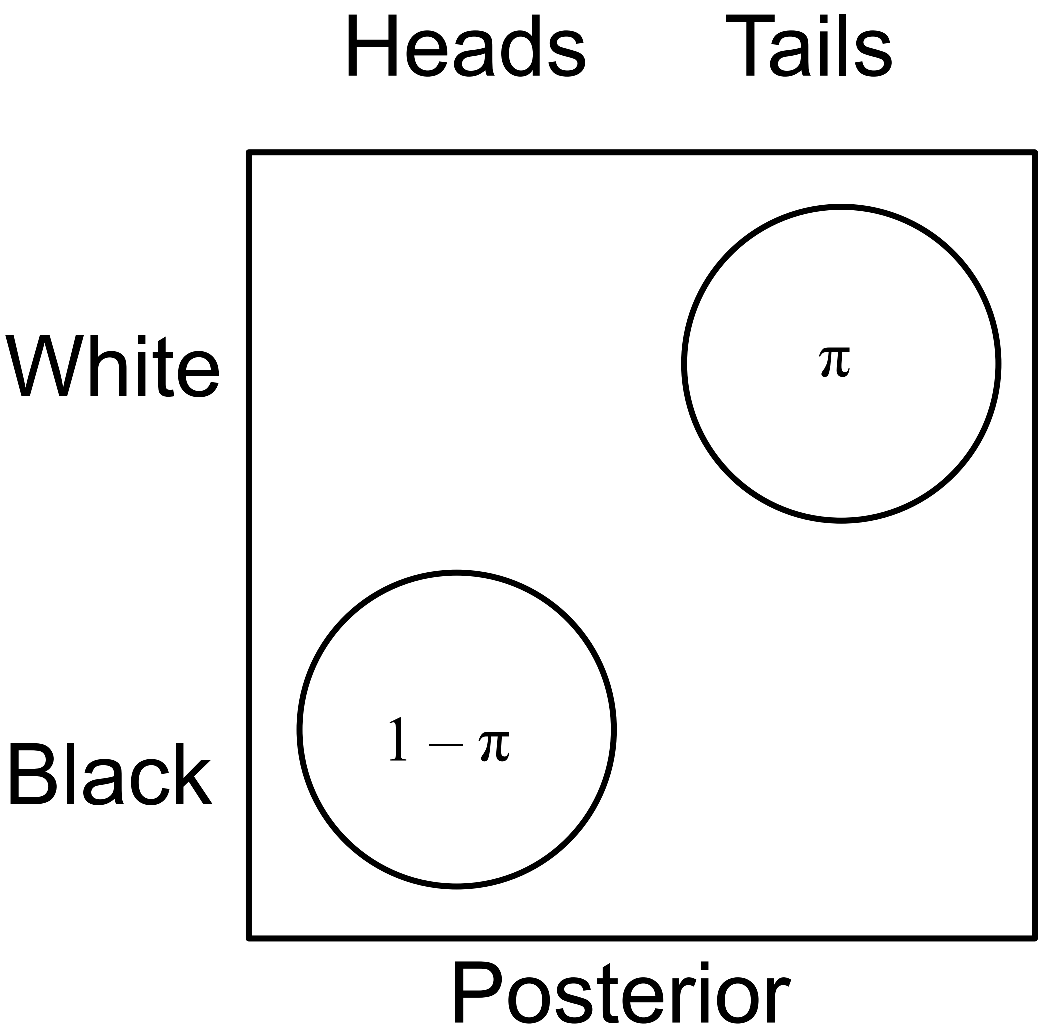
Dempster-Shafer rule

Dempster-Shafer updating yields:

$$\mathfrak{B}^{DS}(A|B)= \mathfrak{B}(A\cup B^{c})-\mathfrak{B}(B^{c})/1-\mathfrak{B}(B^{c})$$

Gelman Paradox: robust Bayesian rule





8. Paradox no.2

Random variable: $X\colon\Omega\rightarrow\mathbb{R}$, which assigns states to prizes.

The probability that X takes on a particular value, say x_i, is given by:

$$\mu_e(X=x_i)=\mu_e(\{\omega\in\Omega: X(\omega)=x_i\})=\mu_e(x_i)$$

$$\mu_e(X\geq x_i)=\mu_e(\{\omega\in\Omega: X(\omega)\geq x_i\})=\mu_e(X_i).$$

A lottery, L, is a vector of length n that assigns a probability to each prize:

$$L\triangleq[\mu_e^L(x_1), \mu_e^L(x_2), \ldots, \mu_e^L(x_i), \ldots, \mu_e^L(x_n)]$$

Behaviour

Decision makers maximize a form of real-valued, rank dependent expected utility:

$$\max_{L\in\mathbb{L}} \varphi\left(\sum_{i=1}^{n} \mu_e^{L}(X_i)\cdot\left(u(\boldsymbol{x}_{\boldsymbol{i}})-u(\boldsymbol{x}_{\boldsymbol{i}-\mathbf{1}})\right)\right)$$

This can be re-arranged and decomposed to the equivalent:

$$\max_{L\in\mathbb{L}} \sum_{i=1}^{n}\left(\alpha.\mathrm{Re}\left(\mu_e^{L}(X_i)\right)+\mathrm{Im}\left(\mu_e^{L}(X_i)\right)\right)\cdot\left(u(\boldsymbol{x}_{\boldsymbol{i}})-u(\boldsymbol{x}_{\boldsymbol{i}-\mathbf{1}})\right)$$

Behaviour

Decision makers satisfy two assumptions:

1. has a real-valued utility function over (lotteries whose payoffs are) money: u

2. converts ambiguous to real utility at a rate:

$$\alpha > 0; \text{ i.e.: } a_L + b_L e \mapsto \alpha a_L + b_L$$

Where: $v(L) = a_L + b_L e$ and $\varphi(v(L)) = \alpha a_L + b_L$

Dynamic behaviour

In dynamic contexts, decision makers satisfy three assumptions:

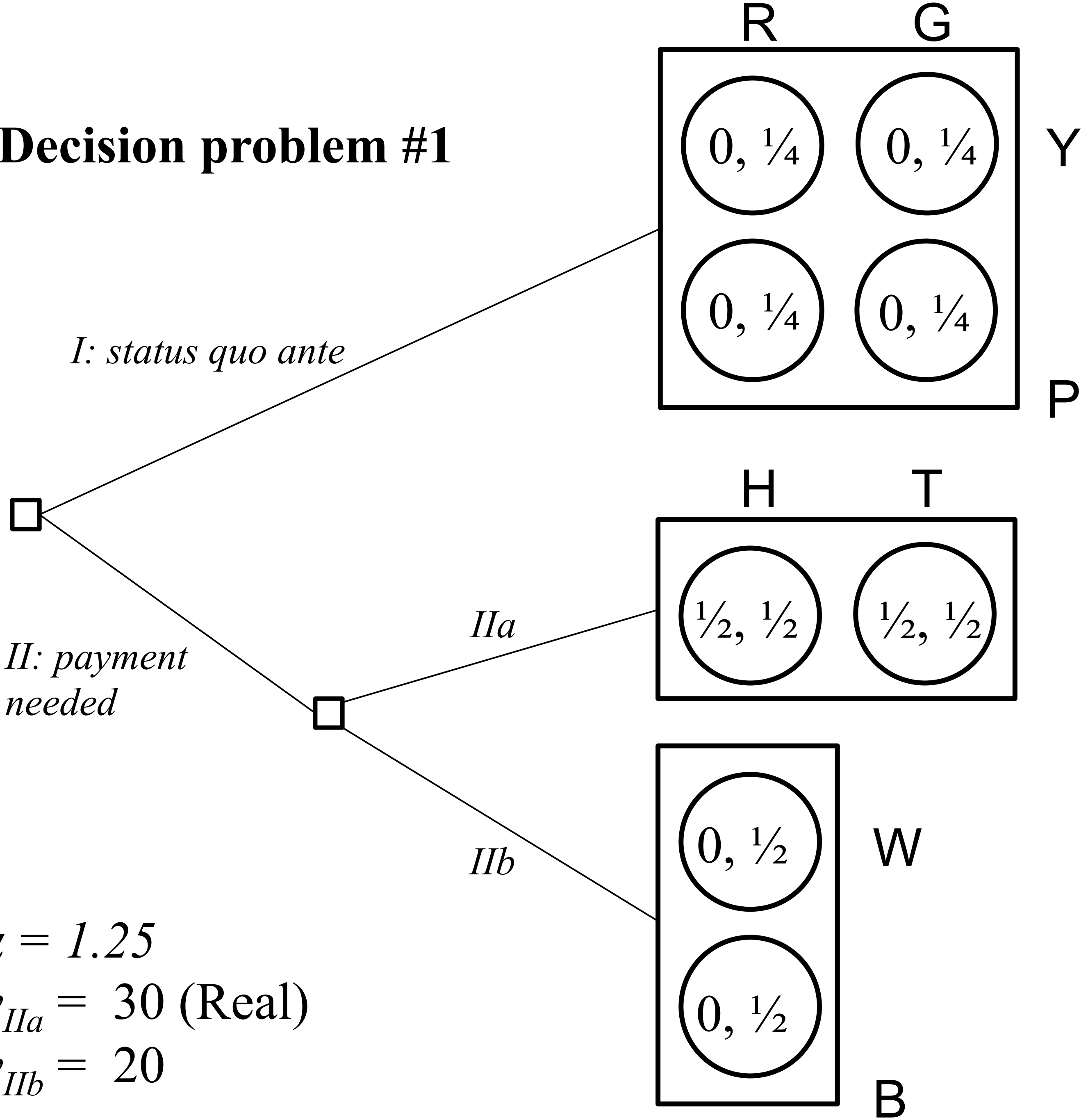
1. beliefs are updated using the full Bayesian rule
2. preferences are constant – i.e., $u(.)$ is constant
3. the maximand is: $\varphi\left(\sum_{i=1}^{n} \mu_e^L(X_i)\cdot\left(u(\mathbf{x}_i)-u(\mathbf{x}_{i-1})\right)\right)$

Gedankenexperiment

	Heads	Tails
White	0, ¼	0, ¼
Black	0, ¼	0, ¼

If you guess correctly which tin contains the Benjamin, you win it!

Decision problem #1



ev = Re | Im | Total

ev = 0 | 25 | 25

ev = 50 | 0 | 50

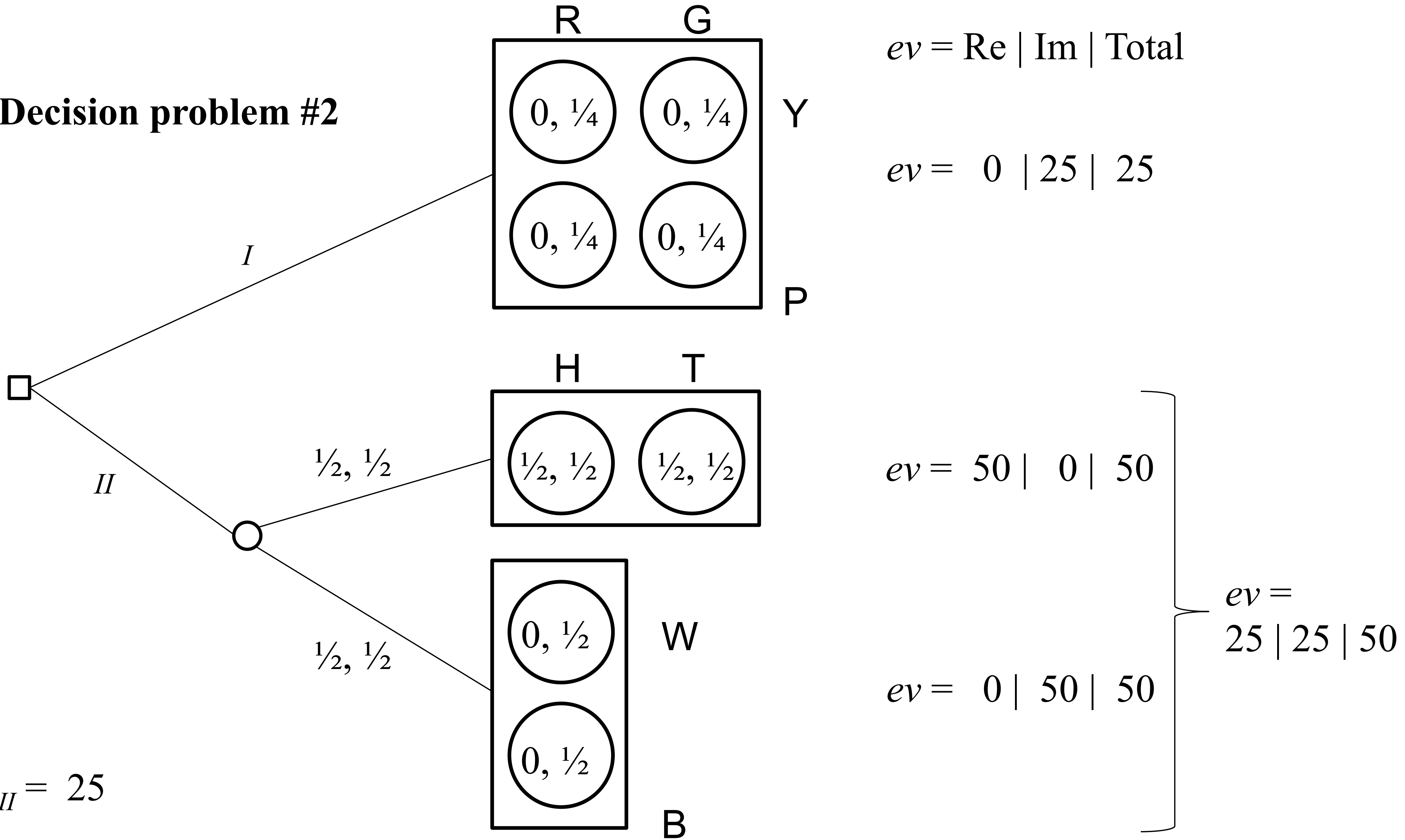
ev = 0 | 50 | 50

If $\alpha = 1.25$

$\Delta ev_{IIa} = 30$ (Real)

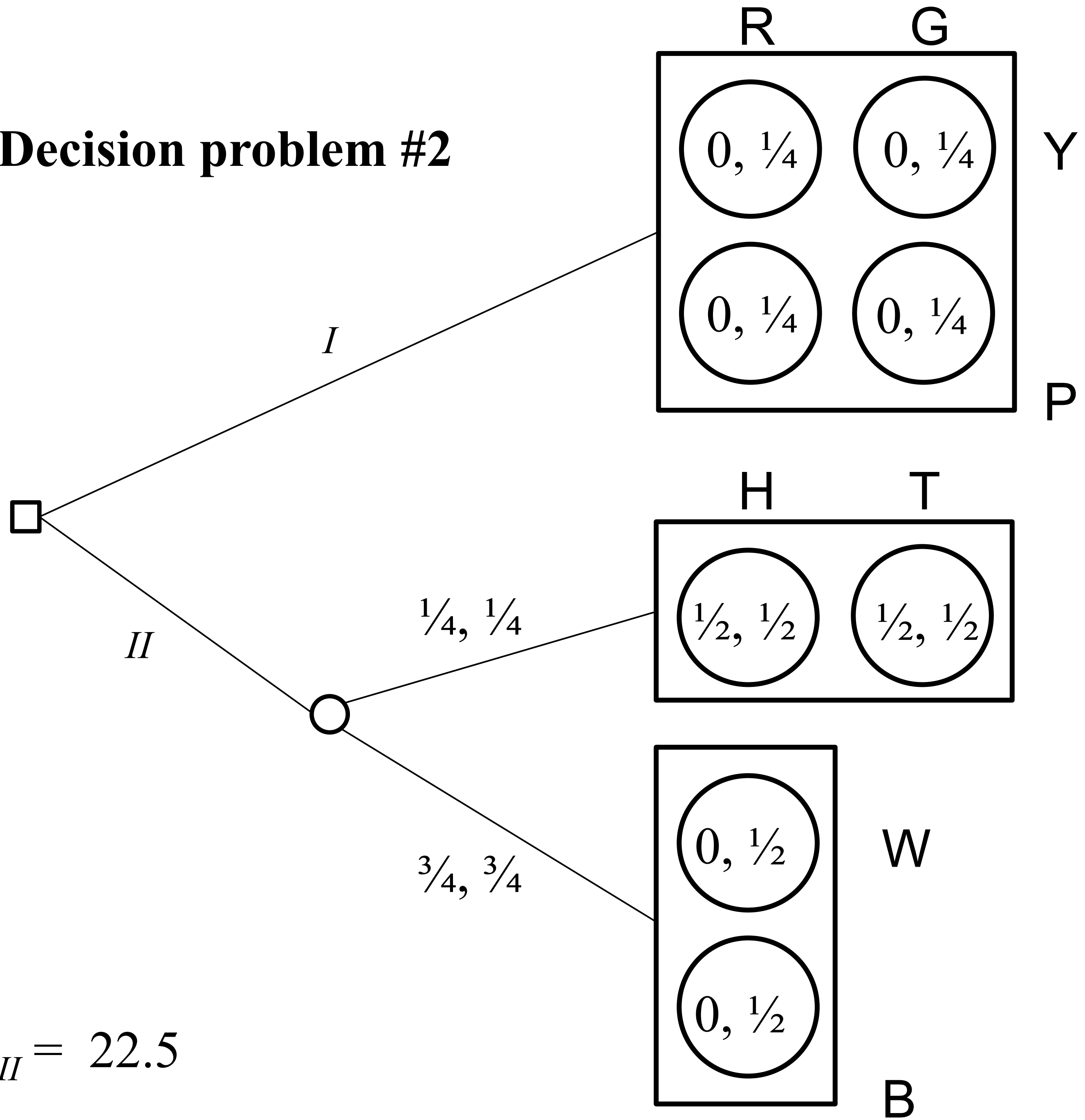
$\Delta ev_{IIb} = 20$

Decision problem #2



$\Delta ev_{II} = 25$

Decision problem #2



ev = Re | Im | Total

ev = 0 | 25 | 25

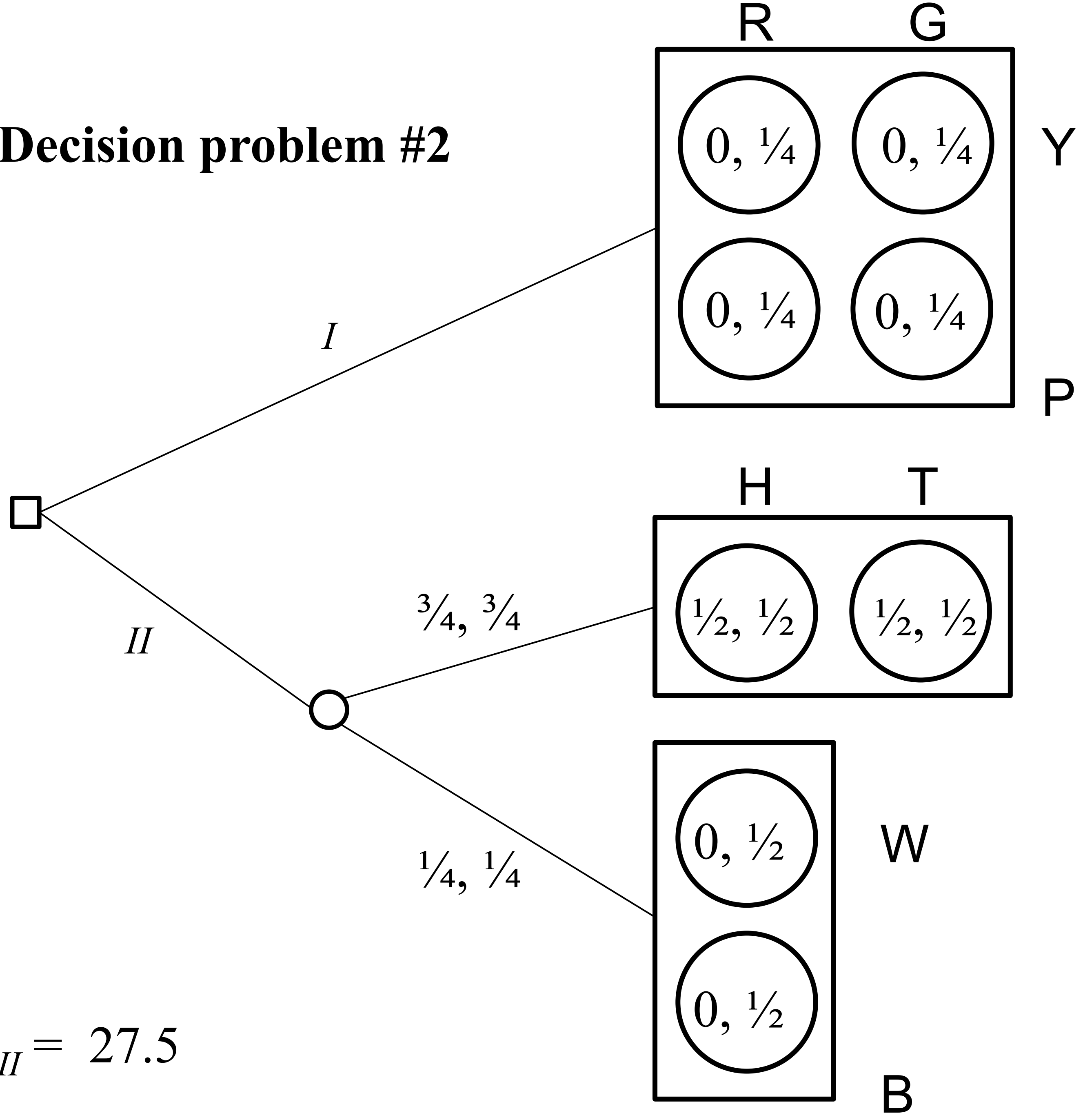
ev = 50 | 0 | 50

ev = 0 | 50 | 50

ev = 12.5 | 37.5 | 50

$\Delta ev_{II} = 22.5$

Decision problem #2



ev = Re | Im | Total

ev = 0 | 25 | 25

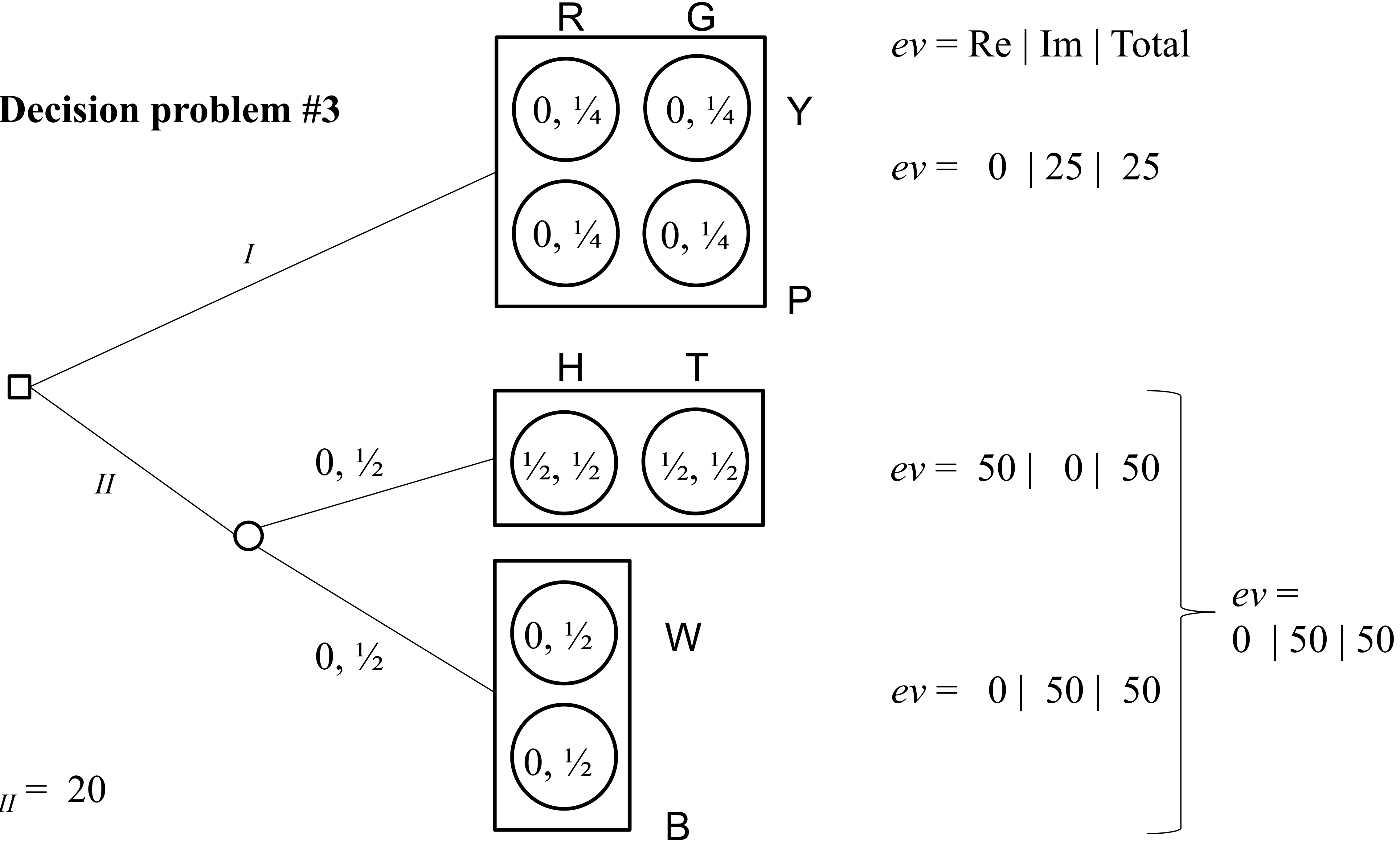
ev = 50 | 0 | 50

ev = 0 | 50 | 50

ev = 37.5 | 12.5 | 50

$\Delta ev_{II} = 27.5$

Decision problem #3



$\Delta ev_{II} = 20$

Conclusions:

The model of updating may be of help in understanding:

1. Keynes' theory of liquidity preference and the 'liquidity trap'

2. Resistance to funding R&D