A perturbation theory analysis of derivatives pricing with transaction costs

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Summary. In this paper I revisit the question of option pricing with transaction costs. For this purpose I study the non-linear Black-Scholes equation derived by Barles and Soner (1998). I concentrate on the asymptotic behavior of the price of the European call options for small transaction costs. Two conclusions emerge from this analysis. First, for small transaction costs the option price as a concave function of the transaction costs with infinite derivative at zero. Second, logarithm of the option price can be approximately written as a sum of the logarithm of the Black-Scoles price and a function of the leverage. The last prediction is corroborated by the data: regression of the logarithm of price on the logarithm of the Black-Scoles price and a measure of leverage explains 94% of variation in the data and the coefficient on the logarithm of the Black-Scoles price is not significantly different from one.

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1 Introduction

The celebrated Black-Scholes formula (Black and Scholes, 1973) provides a way to price a European call option in a market with a frictionless trade populated with fully rational agents. These assumptions are clearly idealizations. Recently, some authors studied option pricing in the presence of transaction costs and trading restrictions. The literature on these issues can be classified into two groups. The first group of papers (see, for example, Leland, 1985 or Boyle and Vorst, 1992) consider the problem of hedging call and put options in the presence of proportional transaction costs, but the strategies chosen in those papers do not meet any optimally conditions. The second group of papers start with assuming that the investors have some utility function and derive the equilibrium option price. This approach is taken by Hodges and Neuberger (1989), Davis, Panas, and Zariphoulou (1993), and Barles and Soner (1998), among others. Virtually no work was done to incorporate bounded rationality into the option pricing theory.[1]

The objective of this paper is to shed some more light on the behavior option prices when the transaction costs are sufficiently small. Barles and Soner's approach is best suited for this purpose, since in their model the

[1] See, however, Basov (2007).

transaction costs and the investor's risk attitudes can be collapsed to a single parameter, a. The price of option follows a non-linear partial differential equation, which is reduced to the Black-Scholes partial differential equation for $a = 0$.

Using perturbation theory analysis I obtain an approximate solution for the option price for the small transaction costs. The first main result is that for a sufficiently small the deviation of the option price from the Black-Scoles price is proportional to $a^{2/3}$, i.e. the option price as a concave function of the transaction costs with infinite derivative at zero. The second main result is that the logarithm of the option price can be approximately written as a sum of the logarithm of the Black-Scoles price and a function of the leverage. A measure of leverage, Δ, is provided in financial data series and tells how much the option prices changes relative to the price of the underlying asset. The last prediction is corroborated by the data: regression of the logarithm of price on the logarithm of the Black-Scoles price and a measure of leverage explains 94% of variation in the data and the coefficient on the logarithm of the Black-Scoles price is not significantly different from one.

The paper is organized in the following way. Section 2 introduces the basics of Barles and Soner's model and the non-linear Black-Scholes equation.

In Section 3 I carry out the perturbation theory analysis of the equation and show that if the underlying stock is not too volatile one can approximately express the logarithm of the option price as a sum of the logarithm of the Black-Scoles price and a function of the leverage. Section 4 describes the data. Section 5 provides the empirical analysis. Section 6 concludes.

2 The model

Consider a market that consists of a risky asset with price $x(t)$ following geometric Brownian motion with drift

$$dx = x[\alpha dt + \sigma dW(t)], \tag{1}$$

and a riskless bond, which pays a compound interest at rate $r > 0$. A European call option is a claim that allows the holder to buy the asset at a pre-specified price, c (known as the strike price) at a pre-specified date T (known as the expiration date). Black and Scholes (1973) have shown that in a market with a frictionless trade populated with fully rational agents the

European call's price, $\psi(x,t)$, solves the following terminal value problem:

$$\begin{cases} \psi_t + rx\psi_x + \frac{\sigma^2}{2}x^2\psi_{xx} = r\psi \\ \psi(x,T) = \max(x-c,0) \end{cases} . \tag{2}$$

Black and Scholes argument relies on the investor's ability to continuously adjust her portfolio by shifting money between the risky asset and the riskless bond. Under these conditions the option can be priced by arbitrage and does not rely on a particular form of utility function.[2] Barles and Soner (1998) modified Black and Scholes analysis by assuming that the agents face a proportional transaction cost, $\mu > 0$, every time they sell their assets. With positive transaction costs it is no longer possible to price the option by arbitrage. Therefore, Barles and Soner assume that the investor has a CARA utility function, i.e. the investor's utility when she possesses wealth w is

$$U(w) = \frac{1-\exp(-\lambda w)}{\lambda}, \tag{3}$$

where $\lambda > 0$ is the investor's Arrow-Pratt's coefficient of absolute risk-aversion.

[2] One still has to assume that the utility function is strictly monotone, i.e. the investor prefers more money to less.

Barles and Soner consider the case when the investor is sufficiently risk averse, while the transaction costs are small. Formally, they proved the following result.

Theorem *Let $\lambda \to \infty$, $\mu \to 0$ and $\mu\sqrt{\lambda} \to a > 0$. Then the European call option price is the unique solution to the following problem:*

$$\left\{ \begin{array}{c} \psi_t + rx\psi_x + \frac{\sigma^2}{2}x^2[1 + S(e^{r(T-t)}a^2x^2\psi_{xx})]\psi_{xx} = r\psi \\ \psi(x,T) = \max(x-c,0) \\ \lim\limits_{x\to\infty} \frac{\psi(x,t)}{x} = 1 \end{array} \right. , \tag{4}$$

where $S(\cdot)$ is the unique solution to the following Cauchy problem:

$$\left\{ \begin{array}{l} \frac{dS}{dz} = \frac{1+S(z)}{2\sqrt{zS(z)}-z} \\ S(0) = 0 \end{array} \right. . \tag{5}$$

Note that the deviation of system (4) from (2) can be captured in a single parameter, a and when $a = 0$ system (4) is reduced to (2). Our next objective is to study the asymptotic of the solution of system (4)-(5) when a is small, i.e. when transaction costs are small comparatively to the appropriate measure of risk-seeking ($1/\sqrt{\lambda}$).

3 Option pricing with small transaction costs: a perturbation theory analysis

Let us consider the behavior of the option price in the case when a, the parameter that captures the deviation of Barles and Soner model from the one of Black and Scholes. For this purpose, let us look for a solution of Cauchy problem (5) in a form:

$$S(z) = bz^{\alpha} + o(z^{\alpha}) \tag{6}$$

for some $\alpha \geq 0$. Substituting (6) into (5) one obtains:

$$\alpha b z^{\alpha-1} + o(z^{\alpha-1}) = \frac{1 + bz^{\alpha} + o(z^{\alpha})}{2\sqrt{b}z^{\frac{1+\alpha}{2}} - z + o(z^{\frac{\alpha+1}{2}})}. \tag{7}$$

First, assume $\alpha \geq 1$. Then equation (7) implies that

$$\alpha b z^{\alpha-1} + o(z^{\alpha-1}) = -z^{-1} + o(z^{\alpha-1}), \tag{8}$$

i.e.

$$\begin{cases} \alpha - 1 = -1 \\ \alpha b = -1 \\ \alpha \geq 1 \end{cases}, \tag{9}$$

which is a contradiction. Therefore, $\alpha < 1$ and equation (7) implies that

$$\alpha b z^{\alpha-1} + o(z^{\alpha-1}) = \frac{1}{2\sqrt{b}} z^{-\frac{1+\alpha}{2}} + o(z^{-\frac{1+\alpha}{2}}), \tag{10}$$

i.e.

$$\begin{cases} \alpha - 1 = -\frac{1+\alpha}{2} \\ \alpha b = \frac{1}{2\sqrt{b}} \\ \alpha < 1 \end{cases} \Longrightarrow \begin{cases} \alpha = \frac{1}{3} \\ b = (\frac{3}{2})^{\frac{2}{3}} \end{cases}. \tag{11}$$

Therefore, for small z function $S(\cdot)$ behaves as

$$S(z) = (\frac{3}{2})^{\frac{2}{3}} z^{1/3} + o(z^{1/3}). \tag{12}$$

To investigate further system (4 let us introduce the following notation:

$$q = \ln x,\ \tau = T - t,\ \varphi(q,\tau) = \ln \psi(e^q, T - \tau),\ \varepsilon = (\frac{3a}{2})^{\frac{2}{3}}. \tag{13}$$

Then function $\varphi(q,\tau)$ solves the following non-linear partial differential equation:

$$-\varphi_\tau + r\varphi_q + \frac{\sigma^2}{2}[\varphi_q^2 + \varphi_{qq} - \varphi_q](1 + \varepsilon e^{\frac{r\tau}{3}}(\varphi_q^2 + \varphi_{qq} - \varphi_q)^{1/3}) = r + o(\varepsilon). \quad (14)$$

Let us look for a solution of equation (14) in a form:

$$\varphi = \varphi_0 + \varepsilon\varphi_1 + o(\varepsilon). \quad (15)$$

Then

$$\varphi_0(q,\tau) = \ln \psi_{BS}(e^q, T-\tau), \quad (16)$$

where $\psi_{BS}(e^q, T-\tau)$ is the Black-Scholes option price, given by

$$\psi_{BS}(x,t) = x\Phi(d_1) - ce^{-rT}\Phi(d_2), \quad (17)$$

where

$$d_1 = \frac{\ln\frac{x}{c} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \quad (18)$$

$$d_2 = d_1 - \sigma\sqrt{T-t}, \quad (19)$$

and φ_1 solves the following linear non-uniform Cauchy problem:

$$\begin{cases} -\varphi_{1\tau} + r\varphi_{1q} + \frac{\sigma^2}{2}[2\varphi_{0q}\varphi_{1q} + \varphi_{1qq} - \varphi_{1q}] = -\frac{\sigma^2}{2}e^{\frac{r\tau}{3}}(\varphi_{0q}^2 + \varphi_{0qq} - \varphi_{0q})^{4/3} \\ \varphi_1(0,q) = 0 \end{cases}. \tag{20}$$

Define linear differential operator $L : C^2(R^2) \to C(R^2)$ by:

$$L(\phi) = -\phi_\tau + r\phi_q + \frac{\sigma^2}{2}[2\varphi_{0q}\phi_q + \phi_{qq} - \phi_q]. \tag{21}$$

The Green function, $G(\tau, q; \zeta, \xi)$ for linear differential L is defined as a solution to the following linear non-uniform Cauchy problem

$$\begin{cases} LG = \delta(\tau - \zeta)\delta(q - \xi) \\ G(0,q) = 0 \end{cases}, \tag{22}$$

where $\delta(\cdot)$ is the Dirac's delta function. Using the Green function, the solution to (20) can be written as:

$$\varphi_{1\tau}(q,\tau) = -\frac{\sigma^2}{2}\iint G(\tau, q; \zeta, \xi)e^{\frac{r\zeta}{3}}(\varphi_{0q}^2(\xi,\zeta) + \varphi_{0qq}(\xi,\zeta) - \varphi_{0q}(\xi,\zeta))^{4/3}d\zeta d\xi. \tag{23}$$

To link expression (23) to the observable financial variables let us introduce a measure of leverage, Δ, by:

$$\Delta = \psi_x. \tag{24}$$

Using (13) and some elementary algebra one can write

$$\varphi_q = \frac{x\Delta}{\psi}. \tag{25}$$

Note that in the main approximation by ε the option price ψ is given by the Black-Scholes formula. Assuming that the volatility of the underlying stock is sufficiently small or the option is long-lived ($rT >> 1$) that formula implies

$$\psi \approx x. \tag{26}$$

Under the same conditions on can neglect the term φ_{0qq} in the integrand and using the mean value theorem obtain

$$\varphi_{1\tau}(q,\tau) = \sigma^2 g(\overline{\Delta}) \tag{27}$$

for some function $g(\cdot)$, where $\overline{\Delta}$ is some average measure of the leverage of the option. Therefore, finally one obtains:

$$\ln \psi(x, t) = \ln \psi_{BS}(x, t) + \varepsilon g(\overline{\Delta}). \tag{28}$$

Formula (28) and definition of ε (see (13)) imply that the first order correction to the Black-Scholes formula is proportional to $a^{2/3}$, i.e. it is concave in transaction costs with an infinite derivative at the origin. Another prediction of formula (28) is that the leverage parameter, Δ, should be used together with the "fair" (Black-Scoles) price to predict the price of an option and that the coefficient on the $\ln \psi_{BS}(x, t)$ is one. In the next two sections I will test the latter prediction using the data on Australian Stock Exchange (ASX) stocks.

4 Data description

The Australian Financial Review (AFR) Market Wrap shares tables cover trading in all main board ASX stocks, using the AFR's own direct data feed from the Australian Stock Exchange. I used data on European call options that include *last sale*, the last price at which the option traded and a set of explanatory variables. The explanatory variables include *fair value*, which is supplied by the ASX and calculated using the Black-Scholes formula. In the absence of the transaction costs this should be the sufficient statistics for the actual price, i.e. all other variables should be statistically insignificant.

The existence of the transaction costs, however, imply that the actual price will deviate systematically from the fair price. To test this prediction we included in the regression a set of other explanatory variables, such as *the exercise price,* i.e. the price at which the option can be exercised, *open interest,* the number of option contracts which are "open" (have not been extinguished by an equal and opposite trade), *implied volatility,* the volatility of the option which is implied by recent quotes at the bid price, *delta,* a measure of leverage, which tells how much the option price will change relative to changes in the underlying asset, and *annual return,* the annualized percentage return to the option writer (seller) from the option premium received, (if the option is held by the taker until expiry), calculated on the current price of the underlying security.

The data in the ASX data set compiled after close of trade 19:28 Tuesday 22/5/2007 contained more than 900 observations. However, after excluding the observations for which some data were missing I was left with 233 data points. The methodology was first to regress the actual trading price for the European stock option on the entire set of available variables and then take a closer look at the dependence of the price of the option on the set variables that turned out to be statistically significant in the first regression. In the next section I describe the results that are in agreement with equation (28).

5 Empirical results

In the first regression I regressed the option price on the entire set of explanatory variable provided by AFR Market Wrap shares tables. The results are shown in Table 1.

Note that only two variables are statistically significant at 5% significance level: fair market price and the leverage parameter, Δ.[3] This result is in an agreement with equation (28). Note, however, that the coefficient before the fair price is significantly different from one and the share of explained variation is relatively small ($R^2 = 0.84$), which points to the functional form misspecification, which is also in agreement with equation (28). Therefore, I

[3] the implied volatility comes close to the significance threshold. It is the only other variable significant at 10% level.

next run a regression

$$\ln \psi(x,t) = \beta_0 + \beta_1 \ln \psi_{BS}(x,t) + \beta_2 \ln \eta + \mu, \tag{29}$$

where

$$\eta = \Delta * vol \tag{30}$$

and vol is the implied volatility, which I use as a proxy for σ^2 in (27). Regression equation (29) is motivated by analytical solution (28). The results are shown in Table 2.

Note that both coefficients β_1 and β_2 are significant, which means that though corrections to Black-Scholes price are small (of the order of 10%) it is no longer a sufficient statistics for the actual price. Despite having less variables the second regression accounts for a bigger share of the total variation. Note also that β_1 is still statistically significantly different from one, but is substantially larger than in the first regression. The difference between β_1 and one can be attributed to relatively short life span of the option, high volatility of the underlying stock, or functional form misspecification.

6 Conclusions

In this paper I analyzed the corrections to the Black-Scholes formula, which arise from the presence of the transaction costs using a perturbation theory approach to a non-linear partial differential equation derived by Barles and Soner. I found an analytical expression for the correction in the main order approximation with a respect to the transaction costs. I found that it takes a particular simple form when written in logarithms rather than in levels. In that case the logarithm of actual price can be approximated as a sum of Black-Scholes price and a function of just single other observable characteristic, which is a combination of the leverage of the option and the volatility of the underlying stock. This conclusion is largely born out by the data. A regression that uses the fair price and the combination leverage and volatility as the explanatory variables explains 94% of variation in the data. Some discrepancies that remain between the theory and the empirical findings (for example, the coefficient on the logarithm fair price is close to, but significantly different from, one) can be explained by relatively short life span of the option, high volatility of the underlying stock, functional form misspecification, or the use of proxies (such as using implied volatility as a proxy for the volatility of the underlying stock).

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Table 1

Dependent variable: last sale

Variable	Coeff	Std	t-stat	lower 95%	upper 95%
Intercept	-0.014	0.176	-0.079	-0.360	0.332
Exercise price	-0.0004	0.003	-0.142	-0.006	0.005
Fair value (B-S)	0.648	0.030	21.4	0.588	0.707
Volume	-0.0002	0.0003	-0.548	-0.0009	0.0005
Open Interest	$1.56 * 10^{-5}$	$3.64 * 10^{-5}$	0.428	$-5.62 * 10^{-5}$	$8.73 * 10^{-5}$
Implied Volatility	-0.010	0.005	-1.880	-0.020	0.0005
Leverage (Δ)	1.130	0.231	4.896	0.675	1.58
Annual % of Return	-0.0003	0.002	-0.189	-0.004	0.003

$R^2 = 0.885$, Adjusted $R^2 = 0.840$

Table 2

Dependent variable: logarithm of last sale

Variable	Coefficient	Std	t-stat	lower 95%	upper 95%
Intercept	-0.001	0.038	-0.031	-0.076	0.074
Log of fair value (B-S)	0.943	0.015	62.49	0.913	0.972
Log(η)	-0.034	0.013	-2.524	-0.061	-0.007

$R^2 = 0.945$, Adjusted $R^2 = 0.944$