IRREDUCIBILITY, TAXES AND THE EXISTENCE OF EQUILIBRIUM

William D. A Bryant*
Department of Economics
MACQUARIE UNIVERSITY NSW 2109
AUSTRALIA

Abstract

The seminal result on the existence of competitive equilibrium in Arrow and Debreu (1954) contains, as one of its conditions, the requirement that each consumer has an endowment which is in the interior of his or her consumption set. The authors assert that such a condition, or something like it, is *necessary* for the existence of competitive equilibrium. They also remark that it is desirable to find a more reasonable condition than that of interior of endowments, particularly in the context of establishing the existence of competitive equilibrium. Irreducibility provides an interesting alternative to the Arrow-Debreu interior endowment condition. Florig (2001) has shown that one form of irreducibility namely, Bergstrom-Florig irreducibility, is indeed necessary for existence (at least when preferences are not price dependent). However, like the interior endowments condition, irreducibility requires that certain *relationships* hold across the economy – relationship which don't have any obvious theoretical motivation and which may well fail in reality. In this paper we show that a breakdown of irreducibility can be repaired by certain tax and transfer schemes, at least as far as the existence of equilibrium is concerned.

Keywords: Irreducibility, quasi-equilibrium, equilibrium, taxes.

JEL Classification: D51

*Contact details: Department of Economics, Macquarie University NSW 2109, Australia. Tel: +61-2-98508465, Fax: +61-2-98508586, email: tbryant@efs.mq.edu.au

1. Introduction

The seminal result on the existence of competitive equilibrium in Arrow and Debreu (1954; Theorem 1) contains, as one of its conditions, the requirement that each consumer has an endowment in the interior of their consumption set. Commenting on this condition, Arrow and Debreu (1954) observe that while it is 'clearly unrealistic'[1] it, or something like it, is actually necessary for the existence of competitive equilibrium.

Research aimed at finding alternatives to 'interior endowments' has spawned a rich list of alternatives. Arrow and Debreu (1954; Theorem 2) introduced 'desireable commodities' and 'productive labour'. Debreu (1962) allows endowments in the consumption set (not necessarily in the interior), along with conditions on the aggregate production set and its relationship to the aggregate consumption set. Arrow and Hahn (1971) present various notions of 'resource relatedness'. Moore (1975) suggests the idea of 'productive consumers'. Danilov and Sotskov (1990) consider 'super self-sufficiency'; Hammond (1993) suggested 'generalised interdependence' and 'no-oligarchy'. Maxfield (1997) develops conditions based on the connectedness of the economy graph. Gale (1957), McKenzie (1959, 1981), Bergstrom (1976, 1991), Florig (2001) Baldry and Ghosal (1999, 2005) present and refine the idea of 'irreducibility'. While these conditions differ in detail, they have something important in common: they are all 'relationship conditions' across the primitives that define the economy. As there is nothing in the operation of an economy that guarantees these relationships will hold, it is of interest to ask if public policy, in particular a tax-transfer scheme, may be able to 'patch' a breakdown in irreducibility if that were to occur. The purpose of this paper is to show

[1] A view shared by many – see for instance Florenzano (2003; pg. 50) and the survey in Bryant (1997).

that a breakdown of irreducibility can be repaired by a tax and transfer scheme, at least as far as the existence of equilibrium is concerned. It is perhaps worth observing that while there are a number of papers which establish the existence of equilibrium with taxes and/or transfers (e.g Mantel (1975), Shafer and Sonnenschein (1976), Dieker and Haller (1990)) in these papers taxes and transfers are regarded as institutional features to be incorporated into an existence argument. The spirit of this paper is that there are circumstances in which equilibrium exists because of taxes and transfers. The paper is organized as follows: Section 2 presents the basic ideas, definitions and major results. Section 3 contains some concluding remarks.

2. Interior endowments, irreducibility and the existence of equilibrium

2.1 Historical background

Duffie and Sonnenschein (1989) give the following interesting account of how the 'interior endowments' condition first came to be used: "Arrow and Debreu began their work on the existence theorem independently. As Arrow writes: 'Debreu and I sent our manuscripts to each other and so discovered our common purpose. We also discovered the same flaw in each others work; we had ignored the possibility of discontinuity when prices vary in such a way that some consumers' income approach zero. We then collaborated, mostly by correspondence, until we had come to a resolution of this problem' (CPII, p. 59). This resolution was to require, in theorem 1 of their paper, that the initial endowment of each household be in the interior of its consumption set." Duffie and Sonnenschein (1989; p. 570).

Having acknowledged that the interior endowments condition was 'clearly unrealistic', and having implicitly expressed the hope that subsequent research would find ways to weaken the condition, Arrow and Debreu (1954) made the following interesting observation: "… the *necessity* of this assumption, or some parallel one, for the validity of the existence theorem points up an important principle; to have equilibrium it is *necessary* that each individual possess some asset or be capable of supplying some labor service which commands a positive price at equilibrium." Arrow and Debreu (1954; p. 270; emphasis added). Thus, whatever condition is discovered to replace interior endowments, it must somehow guarantee that people have a non-zero (indeed survival level) income at equilibrium. One of the most popular alternative conditions to interior endowments is the idea of 'irreducibility'. Mas-Colell, Whinston and Green (1995) hail this class of conditions in the following terms: "Although convenient, [the interior endowment condition cannot] be regarded as extremely weak. It would be unfortunate if the validity of the theory were restricted to [it]. But this is not so: much weaker conditions are available. In particular, McKenzie (1959) has developed a theory of *indecomposable* [irreducible] economies that guarantees that at a quasi-equilibrium the cheaper consumption condition is satisfied for every consumer … The basic idea, informally described, is that an economy is indecomposable [irreducible] if, no matter how we partition the economy into two groups, each of the groups has something for which the other group is willing to exchange something of its own." Mas-Collell, Whinston and Green (1995; p. 633). McKenzie (1987) points out that the role of the condition is as follows: "[irreducibility] guarantees that everyone has income if anyone has income. The meaning of having income is that the consumer is able to reduce his spending at the market price vector below the cost of his allocation and remain within his possible consumption set…" McKenzie (1987; p. 501)[2]. As the counter-examples in Florenzano (1982) show there are non-pathological situations in which irreducibility breaks down. A breakdown in irreducibility must therefore be allowed for and it is then interesting to know if such a breakdown could be repaired by public policy.

2.2 Notation, basic ideas and definitions [3]

Let I be a finite set of consumers, J be a finite set of producers, X_i be the consumption possibility set for consumer i and Y_j be the production possibility set for firm j. The set S of normalized prices is $\{p \in \Re^{\lambda} : \|p\| = 1\}$. The preference correspondence for i is a map $P_i: \prod_{i \in I} X_i \times \prod_{j \in J} Yj \times S \to X_i$ with $P_i(x, y, p) = \{x'_i \in X_i : x'_i \succ_i x_i\}$ and $x_i \notin P_i(x, y, p)$.

Also let $\hat{P}_i(x, y, p) = \{x'_i \in X_i : x'_i = x_i + \lambda(x''_i - x_i), 0 < \lambda \leq 1, x''_i \in P_i(x, y, p)\}$ be the 'augumented preference correspondence' for i. θ_{ij} is the share of firm j's profit going to consumer i and consumer i's endowment is $\omega_i \in \Re^{\lambda}$. The total wealth of consumer i at p is $w_i = p.\omega_i + \sum_{j \in J} \theta_{ij} p.y_j$. The disposal cone Z, is a convex cone with vertex $0 \in Z$ and contained in $-\Re_+^{\lambda}$. $Z^0 = \{p \in \Re^{\lambda}: p.z \leq 0 \ \forall z \in Z\}$ is the polar cone of Z. The set of attainable allocations for $\mathcal{E}$ is $A(\mathcal{E}) = \{(x, y) \in \prod_{i \in I} X_i \times \prod_{j \in J} Yj : \sum_{i \in I} x_i - \sum_{j \in J} y_j - \omega \in Z\}$.

$\hat{X} = \{x \in \prod_{i \in I} X_i : \exists y \in Y \text{ and } \sum_{i \in I} x_i - y - \omega \in Z\}$ are the attainable consumption

[2] See also McKenzie (2003) for further discussion of the condition.

[3] We follow the notation and definitions in Florenzano (2003).

allocations and $\hat{Y} = \{y \in Y : \exists x \in \prod_{i \in I} X_i$ and $\sum_{i \in I} x_i - y - \omega \in Z\}$ is the attainable total production set.

2.3 Quasi-equilibrium, equilibrium and some irreducibility notions

A standard way to prove the existence of equilibrium is to firstly prove the existence of a quasi-equilibrium and to then observe that a quasi-equilibrium is a (full) equilibrium if each consumer has access to a cheaper point relative to the quasi-equilibrium allocation.

DEFINITION 2.1 An allocation $(x, y) \in \prod_{i \in I} X_i \times \prod_{j \in J} Yj$ is called *attainable* if $\sum_{i \in I} x_i - \sum_{j \in J} y_j - \omega \in Z$.

DEFINITION 2.2 A tuple (x*, y*, p*) consisting of an attainable allocation (x*, y*) and a nonzero price vector p* is called a *quasi-equilibrium* if: (i) for every $i \in I$, $p^*x_i^* \leq w_i^*$ and $x_i \in P_i(x^*, y^*, p^*)$ implies $p^*x_i \geq p^*x_i^*$; (ii) for each $j \in J$ and $y_j \in Y_j$, y^*_j is profit maximizing at p* so $\forall\, y_j \in Y_j$ $p^*y_j \leq p^*y_j^*$; (iii) $p^* \in Z^0$ and $p^*.\sum_{i \in I} x_i^* = p^*.\sum_{j \in J} y_j^* + p^*.\sum_{i \in I} \omega_i$. A quasi-equilibrium is *non-trivial* if $\exists\, i \in I$ and $x_i \in X_i$ such that $p^*x_i < p^*x_i^*$. Let $\delta_i(p^*) = \{x_i \in X_i : p^*x_i < p^*x_i^* = p^*\omega_i + \sum_{j \in J} \theta_{ij} p^*y_j\}$ be the set of 'cheaper points' relative to the allocation that consumer i receives in the quasi-equilibrium.

Florenzano (2003; p. 56) specifies conditions on the primitives of the economy that ensure the existence of a quasi-equilibrium. However, as is well known, a quasi-equilibrium is of limited interest as an equilibrium concept for an economy. For instance a quasi-equilibrium allocation need not be utility maximizing for consumers. For that reason we define the idea of a (full) equilibrium.

DEFINITION 2.3 A tuple (x*, y*, p*) consisting of an attainable allocation (x*, y*) and a nonzero price vector p* is called an *equilibrium* if: (i) for every $i \in I$, $p^*x_i^* \leq w_i^*$ and $x_i \in P_i(x^*, y^*, p^*)$ implies $p^*x_i > p^*x_i^*$; (ii) for each $j \in J$ and $y_j \in Y_j$, y^*_j is profit maximizing at p* so $\forall y_j \in Y_j$ $p^*y_j \leq p^*y_j^*$; (iii) the cost of the disposal needed to achieve equilibrium is zero so that $p^* \in Z^0$ and $p^*.\sum_{i \in I} x_i^* = p^*.\sum_{j \in J} y_j^* + p^*.\sum_{i \in I} \omega_i$.

In order to pass from the existence of a quasi-equilibrium to the existence of an equilibrium each consumer needs access to a 'cheaper point' relative to the consumption vector they are allocated in the quasi-equilibrium (see for instance the discussion in Florenzano (2003; p. 62). This is equivalent to the requirement that $\delta_i(p^*) \neq \emptyset$ for each consumer i. One way to ensure that $\delta_i(p^*) \neq \emptyset$ for all i, is to assume that every consumer has an endowment in the interior of their consumption set. Alternatively it might be assumed that the economy satisfies some sort of irreducibility condition.

DEFINITION 2.4: Let $T_y(Y) = cl\{z \in \Re^{\lambda}: z = \lambda(y' - y), \lambda > 0, y' \in Y, y = \sum_{j \in J} y_j\}$ then an economy $\mathcal{E}$ is *McKenzie-Debreu irreducible* if, for any partition of I into two non-empty sub-sets $\{I_1, I_2\}$and for each $(x, y, p) \in A(\mathcal{E}) \times (S \cap Z^0)$, there exists $x' \in \prod_{i \in I} X_i$ such that:

(i) $x' \in cl(\hat{P}_i(x, y, p))$ for each $i \in I_1$ with for some $i_1 \in I_1$, $x'_{i1} \in \hat{P}_{i1}(x, y, p)$;

(ii) $\sum_{i \in I1} (x'_i - x_i) + \sum_{i \in I2} (x'_i - \omega_i) = y \in (T_{\underline{y}}(Y) + Z)$

REMARK: As Forenzano (2003; 63) puts it: "… $\mathcal{E}$ is McKenzie-Debreu irreducible if for any partition $\{I_1, I_2\}$of the set of consumers into two nonempty subsets and for any couple of an attainable allocation $(x, y) \in A(\mathcal{E})$ and of a price vector $p \in S \cap Z^0$, the group I_1 (after possible disposal) may be moved to a preferred position, as expressed by (i), by adding a vector of the tangent cone of Y at $\underline{y}$ plus a feasible trade from I_2".

DEFINITION 2.5 An economy $\mathcal{E}$ is *Bergstrom-Florig irreducible* if for any partition of I into two non-empty sub-sets $\{I_1, I_2\}$and for each $(x, y, p) \in A(\mathcal{E}) \times (S \cap Z^0)$, there exist real numbers $\theta_i > 0$, $i \in I$ and $x' \in \prod_{i \in I} X_i$ such that:

(i) $x' \in cl(\hat{P}_i(x, y, p))$ for each $i \in I_1$ with for some $i_1 \in I_1$, $x'_{i1} \in \hat{P}_{i1}(x, y, p)$;

(ii) $\sum_{i \in I} \theta_i (x'_i - \omega_i - \sum_{j \in J} \theta_{ij} y_j) \in (T_{\underline{y}}(Y) + Z)$

As inspection of (ii) in the DEFINITION 2.4 makes clear, the existence of a feasible trade from the group of consumers I_2 (a set which may contain just one element), depends, to some extent, on the location of the ω_i's relative to the x'_i's. To see this note clause (ii) can be rewritten as:

$$\sum_{i\in I2} \omega_i = \sum_{i\in I} x'_i - y - \sum_{i\in I1} x_i \qquad \text{for } y \in (T_{\underline{y}}(Y) + Z) \tag{1}$$

As the expression in (1) makes clear, to get McKenzie-Debreu irreducibility a relationship needs to hold between the endowments of the consumers in group I_2 and the preferences of the consumers in group I_1. It is not obvious that for a given economy, such a relationship need hold (again recall the counter-examples in Florenzano (1982). Suppose that McKenzie-Debreu irreducibility does not naturally hold, can some form of public policy repair it?

DEFINITION 2.6 A *tax-transfer scheme that induces McKenzie-Debreu irreducibility* is a scheme that allocates to consumer i the share $t_i\omega$ of the total endowment ω, such that the following holds. For any partition of I into sub-sets $\{I_1, I_2\}$such that $I_1 \cap I_2 = \varnothing$ and $I_1 \cup I_2 = I$ and for each $(x, y, p) \in A(\mathcal{E}) \times (S \cap Z^0)$, there exists $x' \in \prod_{i\in I} X_i$ such that:

(i) $cl(\hat{P}_i(x, y, p))$ for each $i \in I_1$ with for some $i_1 \in I_1$, $x'_{i1} \in \hat{P}_{i1}(x, y, p)$;

(ii) $\sum_{i\in I1} (x'_i - x_i) + \sum_{i\in I2} (x'_i - t_i\omega) = y \in (T_{\underline{y}}(Y) + Z)$

PROPOSITION: *If (x^*, y^*, p^*) is a non-trivial quasi-equilibrium for $\mathcal{E}$* then (x^*, y^*, p^*) is also an equilibrium for $\mathcal{E}$ i*f:*

(i) *each X_i and Y_j are convex;*

(ii) *for all $i \in$* I and $(x, y, p) \in A(\mathcal{E}) \times (S \cap Z^0)$ if $z_i \in P_i(x, y, p)$ and $v_i \in X_i$ then there exists $0 < \lambda \leq 1$ such that $(\lambda v_i + (1 - \lambda)z_i) \in P_i(x, y, p)$;

(iii) *$\mathcal{E}$ is tax-induced McKenzie-Debreu irreducible*.

PROOF: The proof is given in the APPENDIX.

REMARK: Inspection of the condition for Bergstrom-Florig irreducibility indicates that similar arguments can be made in that case also. This is interesting because as Florig (2001) shows, Bergstrom-Florig irreducibility is necessary for $\delta_i(p^*) \neq \emptyset$ provided that every quasi-equilibrium is non-trivial and preferences do not depend on prices.

3. Conclusion

The seminal result on the existence of competitive equilibrium in Arrow and Debreu (1954) contains, as one of its conditions, the requirement that each consumer has an endowment which is in the interior of his or her consumption set. Various notions of irreducibility provide interesting alternatives to that implausible condition. However, like the interior endowments condition, irreducibility requires that certain relationships hold across the primitives that define the economy. As there is no clear mechanism at work in the economy to ensure that such conditions hold, they may fail in reality. In this paper we show that a breakdown of McKenzie-Debreu irreducibility can be repaired by a simple tax and transfer scheme, at least as far as the existence of equilibrium is concerned. It is of interest to know if existence results of this form can be proved using alternative tax-transfer schemes. This is a subject for future research.

APPENDIX

Proof of Proposition: It is therefore enough to show that if $\mathcal{E}$ is tax-induced McKenzie-Debreu irreducible then it is Bergstrom-Florig irreducible. Let {I, I_2} be a partition of I into non-empty subsets and (x, y, p) $\in$ A($\mathcal{E}$) $\times$ (S $\cap$ Z^0). Assume that x' $\in \prod_{i \in I} X_i$ satisfies for the partition {I_1, I_2} the conditions of DEFINITION 2.6 so that the economy is McKenzie-Debreu irreducible. From the relation

$$\sum_{i \in I1} (x'_i - x_i) + \sum_{i \in I2} (x'_i - x_i) + \sum_{i \in I2} (x_i - t_i\omega) \in (T_{\underline{y}}(Y) + Z)$$

we get that $\sum_{i \in I} \theta_i (x'_i - t_i\omega - \sum_{j \in J} \theta_{ij}y_j) + \sum_{i \in I2} (x_i - t_i\omega) \in (T_{\underline{y}}(Y) + Z)$, this implies

$\sum_{i \in I} \theta_i (x'_i - t_i\omega - \sum_{j \in J} \theta_{ij}y_j) + 2\sum_{i \in I2} \{(x'_i + x_i)/2 - t_i\omega - \sum_{j \in J} \theta_{ij}y_j\} \in (\{- \sum_{i \in I2} \sum_{j \in J} \theta_{ij}y_j\} +$ $(T_{\underline{y}}(Y) + Z)$. Since $- \sum_{i \in I2} \sum_{j \in J} \theta_{ij}y_j \in T_{\underline{y}}(Y)$ the convexity of each X_i yields condition (ii) in DEFINITION 2.5 i.e. the economy is also Bergstrom-Florig irreducible. PROPOSITION 2.3.3 in Florenzano (2003; p. 65) establishes that if $\mathcal{E}$ is Bergstrom-Florig irreducible then a under the conditions assumed in our PROPOSITION every non-trivial quasi-equilibrium is an equilibrium, which completes the proof. ■

REFERENCES

K. J. Arrow and G. Debreu (1954), "Existence of equilibrium for a competitive economy". *Econometrica*, 22, 265 – 290.

K. J. Arrow and F. H. Hahn (1971), *General Competitive Analysis*. Holden-Day, San Francisco.

R. Baldry and S. Ghosal (1999), "Irreducibility in exchange economies". Paper presented at 12th European General Equilibrium Theory Workshop in Copenhagen, 4 – 6 June 1999.

R. Baldry and S. Ghosal (2005), "Irreducible economies and strongly connected graphs". *Journal of Mathematical Economics*, 41, 937 – 956.

T. C. Bergstrom (1976), "How to discard 'free disposability' – at no cost". *Journal of Mathematical Economics*, 3, 131 – 134.
Bergstrom (1991),

W. D. Bryant (1997), "Conditions for the existence of market equilibrium". *Journal of Economic Education*, 12, 230 – 254.

V. I. Danilov and A. I. Sotskov (1990), "A generalized economic equilibrium". *Journal of Mathematical Economics*, 19, 341 – 356.

G. Debreu (1962), "New concepts and techniques for equilibrium analysis". *International Economic Review*, 3, 257 – 272.

E. Dieker and H. Haller (1990), "Tax systems and direct mechanisms in large finite economies", *Journal of Economics*, 52, 99 – 116.

D. Duffie and H. Sonnenschein (1989), "Arrow and general equilibrium theory". *Journal of Economic Literature*, 27, 565 – 598.

M. Florenzano (1982),"The Gale-Nikaido-Debreu lemma and the existence of transitive equilibrium with or without the free disposal assumption". *Journal of Mathematical Economics*, 9, 113 – 134.

M. Florenzano (2003), *General Equilibrium Analysis: Existence and Optimality Properties of Equilibria*. Kluwer Academic Publishers, Boston.

M. Florig (2001), "On irreducible economies", *Annales d'Économie et de Statistiques*, 61, 183 – 199.

D. Gale (1957), "Price equilibrium for linear models of exchange". Technical Report P-1156, The Rand Corporation.

P. J. Hammond (1993), “Irreducibility, resource relatedness and survival in equilibrium with individual non-convexities”. In R. Becker, M. Boldrin and W. Thomson editors, *General Equilibrium, Growth and Trade*, pp. 73 – 115. Academic Press, San Diego.

R. Mantel (1975), “General equilibrium and optimal taxes”, *Journal of Mathematical Economics*, 2, 187 – 200.

A. Mas-Colell, M. D. Whinston and J. Green (1995), *Microeconomic Theory*. Oxford University Press, Oxford.

R. R. Maxfield (1997), “General equilibrium theory and the theory of directed graphs”. *Journal of Mathematical Economics*, 27, 23 – 51.

L. McKenzie (1959), “On the existence of general equilibrium for a competitive market”. *Econometrica*, 27, 54 – 71.

L. McKenzie (1981), “The classical theorem on the existence of competitive equilibrium”. *Econometrica*, 49, 819 – 841.

J. Moore (1975), “The existence of ‘compensated equilibrium’ and the structure of the Pareto efficiency frontier”. *International Economic Review*,16, 267 – 300.

W. Shafer and H. Sonnenschein (1976), “Equilibrium with externalities, commodity taxation and lump sum transfers”. *International Economic Review*, 17, 601 – 611.