The Law of Demand: A 'Revealed Preference' Approach*

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March 14, 2016

Abstract

Milleron and Malinvaud (1974) and Mityushin and Polterovich (1978) provide conditions under which the Uncompensated Law of Demand (ULD) holds. Their conditions are based on the properties of a consumer's (unobservable) preference relation/utility function. In this paper we introduce the idea of a 'Giffen commodity vector' and show how, in terms of it and the derivative of a consumer's inverse demand, a new necessary and sufficient for the ULD to hold. As both the Giffen vector and a consumer's inverse demand are observable, this allows to obtain a 'Revealed Preference' style necessary and sufficient condition for the ULD without having to consult the M-P sufficient, or necessary and sufficient conditions on unobservables.

JEL-Classification: D1, D5
Keywords: Law of demand, monotone, necessary and sufficient conditions, Giffen vector, inverse demand.

1 Introduction

The Uncompensated Law of Demand (ULD) is perhaps the Holy Grail of Economics[1]. Roughly stated, the ULD says that demand by a consumer is decreasing (or at least non-increasing), in price. Conditions under which the ULD holds have been investigated by numerous workers, including Milleron and Malinvaud (1974), Mityushin and Polterovich (1978)[2], Kannai and Selden (2014), Mas-Colell (1991) and Quah (2000, 2003). Milleron and Malinvaud (1974) and Mas-Colell (1991) propose conditions under which the ULD holds in the case of homothetic preferences and Quah (2003) for separable preferences. For general preferences, Mityushin and Polterovich (1978) and independently Milleron and Malinvaud (1974), provide conditions for the ULD to hold for a given preference relation that are expressed as restrictions on its representing, real valued utility function.

All the conditions currently available in the literature put restrictions on consumer data such as preferences or utility. Such data are typically unobservable. In this paper we therefore: (i) introduces the idea of a 'Giffen commodity vector', z^*; (ii) show that z^* has an interesting geometric interpretation; and (iii) that the ULD holds if and only if an expression involving z^* and the derivative of a consumer's inverse demand function is less (both of which are in principle observable), than 0. This allows to obtain directly a necessary and sufficient condition for the ULD without having to go through the preference based M-P style sufficient, or necessary and sufficient conditions[3]. Our main contribution is then a condition for the ULD that is Revealed Preference in style.

*We are very grateful to an anonymous referee for numerous suggestions, which helped us recast this paper and the Editor for allowing this submission at no financial cost to us.

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[1] Similar remarks may be found at numerous points in the literature (see for instance Katzner (1970, pg. 59); Barten and Böhm (1982, pg. 417); Chiang (1984, pp. 407–408); Luenberger (1995, p. 151); Eaton et al. (2002, pg. 123).)

[2] There doesn't appear to be any agreement on the spelling of the name of the first author here. For instance, Mas-Colell (1991, p. 112) render it 'Mitiushin', while Kannai and Selden (2014, p. 2) have it as 'Mitjushin'. The English translation of the original Russian version of the M-P paper, recorded on Polterovich's website, has it as 'Mityushin' - see http://mathecon.cemi.rssi.ru/vm_polterovich/files/MitPoltDemand_engl.pdf

[3] While writing this paper we became aware of the examples of strictly-concave, non-spliced, Giffen-compatible utility function, given in Biederman (2015). However, these examples arises from an examination of elasticities of the consumer's marginal rate of substitution with respect to commodity consumption while the examples herein arose from an examination of the inequalities of Mityushin and Polterovich.

2 Preliminaries

Commodities are dated, located (and possibly state contingent) bundles of characteristics. The *commodity space* is $\mathbb{R}^L_+$ and the consumer's *consumption possibility* set $X \subseteq \mathbb{R}^L_+$ is a non-empty, convex subset of the commodity space. A strictly convex and concavifiable binary *preference relation*, $\preceq$ is defined on X. Consumer *demand* is a map $x\colon \mathbb{R}^L_{++} \times \mathbb{R}_+ \to X$, denoted by $x(p,w)$, and income w is normalized so that $p^T x(p,w) = 1$. With slight abuse of notation we let $x(p,1) \equiv x(p)$. If $(p'-p)^T(x(p') - x(p)) < 0$ for any $p' \neq p$ then demand is *monotone* (or the ULD holds).[4]

Assumption 1. The preference relation $\preceq$ admits representation by a utility function $u : X \to \mathbb{R}$, where $u(x)$ is C^3, strictly monotone and concave.

Thus, if

$$\mathrm{D}u(x) = \begin{bmatrix} \frac{\partial u(x)}{\partial x_1} \\ \vdots \\ \frac{\partial u(x)}{\partial x_L} \end{bmatrix} \text{ and } \mathrm{D}^2 u(x) = \begin{bmatrix} \frac{\partial^2 u(x)}{\partial x_1^2} & \cdots & \frac{\partial^2 u(x)}{\partial x_L \partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 u(x)}{\partial x_1 \partial x_L} & \cdots & \frac{\partial^2 u(x)}{\partial x_L^2} \end{bmatrix}$$

denote the vector of partial derivatives and the Hessian matrix of second derivatives, then $\mathrm{D}u(x) > 0$ and $\mathrm{D}^2 u(x)$ is negative definite.

Assumption 2. The preference relation $\preceq$ generates a differentiable demand function $x(p)$.

In the case where demand is differentiable, monotonicity of demand at a price vector p holds if for any non-zero price change in $\mathbb{R}^L$:

$$\sum_{\ell,k=1}^{L} \frac{\partial x_\ell}{\partial p_k} \Delta p_\ell \Delta p_k < 0 \tag{1}$$

3 The Milleron–Mityuschin–Polterovich result

The Milleron–Mityuschin–Polterovich result says $x(p)$ is monotone at p, if and only if for $x = x(p)$:

$$\frac{x^T \mathrm{D}u}{\mathrm{D}u^T \left(\mathrm{D}^2 u\right)^{-1} \mathrm{D}u} - \frac{x^T \mathrm{D}^2 u \; x}{x^T \mathrm{D}u} < 4 \tag{2}$$

If the Hessian matrix $\mathrm{D}^2 u(x)$ in (2) is singular, then following Kannai and Selden (2014, footnote 6, p. 4) its inverse is thought of as having infinite eigenvalues which correspond to zero eigenvalues of the Hessian. If they appear in the denominator of the first term of (2), then this term is 0.

Although not explicit in Mityushin and Polterovich (1978), equation (2) is derived by maximizing $Z = z^T \mathrm{D}_x g z$ over $z \in \mathbb{R}^L$ satisfying $z^T \mathrm{D}u = x^T \mathrm{D}u$. Here $g\colon \mathbb{R}^L_+ \to \mathbb{R}^L_+$ is the *inverse demand function* $g(x) = \frac{w}{x^T \mathrm{D}u} \mathrm{D}u$, and $x \in \mathbb{R}^L_+$ is fixed. Mityushin and Polterovich note that

$$z^* = \frac{\frac{1}{2} x^T \mathrm{D}u}{\mathrm{D}u^T \left(\mathrm{D}^2 u\right)^{-1} \mathrm{D}u} \left(\mathrm{D}^2 u\right)^{-1} \mathrm{D}u + \frac{1}{2} x, \tag{3}$$

maximizes Z and that the maximum is given by,

$$Z^* = \frac{w}{4} \left[\frac{x^T \mathrm{D}u}{\mathrm{D}u^T \left(\mathrm{D}^2 u\right)^{-1} \mathrm{D}u} - \frac{x^T \mathrm{D}^2 u \; x}{x^T \mathrm{D}u} - 4 \right]. \tag{4}$$

For completeness we include derivations of equations (3) and (4). Accordingly, suppose z^* maximizes $Z = z^T \mathrm{D}_x g z$ over $z \in \mathbb{R}^L$ satisfying $z^T \mathrm{D}u = x^T \mathrm{D}u$. For simplicity we use the assumption that income is normalized so $w = 1$. The constraint ensures that

$$z^{*T} \mathrm{D}_x g z^* = z^{*T} \left[\frac{\mathrm{D}^2 u}{x^T \mathrm{D}u} - \frac{\mathrm{D}u \mathrm{D}u^{\mathrm{T}}}{\left(x^T \mathrm{D}u\right)^2} - \frac{\mathrm{D}u \left(\mathrm{D}^2 u \; x\right)^{\mathrm{T}}}{\left(x^T \mathrm{D}u\right)^2} \right] z^* = \frac{z^{*T} \mathrm{D}^2 u z^*}{x^T \mathrm{D}u} - 1 - \frac{x^T \mathrm{D}^2 u z^*}{x^T \mathrm{D}u}$$

[4] This is the definition in Kannai and Selden (2014, p. 3). The definition in Mas-Colell et al. (1995, p. 111) is that consumer demand is monotone if $(p'-p)[x(p',w) - x(p,w)] \leq 0$ for any p, p', w with strict inequality if $x(p',w) \neq x(p,w)$.

The first order conditions that z^* satisfies are given by,

$$\lambda \mathrm{D}u = 2\frac{\mathrm{D}^2 u z^*}{x^T \mathrm{D}u} - \frac{\mathrm{D}^2 u x}{x^T \mathrm{D}u} \text{ and } {z^*}^T \mathrm{D}u = x^T \mathrm{D}u$$

The first equality here implies that $z^* = \frac{1}{2}\left[\lambda x^T \mathrm{D}u \big(\mathrm{D}^2 u\big)^{-1} \mathrm{D}u + x\right]$ and the second gives $\lambda = \left[\mathrm{D}u^T \big(\mathrm{D}^2 u\big)^{-1} \mathrm{D}u\right]^{-1}$. Thus,

$$z^* = \frac{1}{2}\left[\frac{x^T \mathrm{D}u}{\mathrm{D}u^T \big(\mathrm{D}^2 u\big)^{-1} \mathrm{D}u}\big(\mathrm{D}^2 u\big)^{-1} \mathrm{D}u + x\right]$$

Prices are given by $p = g(x) = \frac{w}{x^T \mathrm{D}u}\mathrm{D}u$ and income by $w = p^T x$. So with $y = \big(\mathrm{D}^2 u\big)^{-1}\mathrm{D}u$,

$$z^* = \frac{w}{2}\left[\frac{y}{p^T y} + \frac{x}{p^T x}\right]. \tag{5}$$

We refer then to the vector z^* given in equation (5) as the *Giffen commodity vector*. In order to establish our first result we need:

Definition 1. [Income expansion path; Varian (1992, p. 116)]: If prices are held fixed and a consumer's income is allowed to vary, then the resulting locus of utility maximizing consumption bundles is the consumer's *income expansion path*.

Lemma 3.1. *Suppose u is a utility function which determines preferences which satisfy Assumption 1 and Assumption 2. If $x \in \mathbb{R}^\ell_+$ maximizes u with prices given by $p \in \mathbb{R}^\ell_{++}$ then the vector $y = \big(\mathrm{D}^2 u\big)^{-1}\mathrm{D}u$ is tangent to the income expansion path determined by u, p and x.*

Proof. To see that y is tangent, let $\gamma(t)$ be a smooth parametrisation of the income-offer curve. Thus,

$$\mathrm{D}u(\gamma(t)) = \lambda(t)\mathrm{D}u,$$

where $\lambda(t)$ is a smooth real valued function. Differentiating with respect to t gives,

$$\mathrm{D}^2 u \gamma'(t) = \lambda'(t)\mathrm{D}u \implies \gamma'(t) = \lambda'(t)\big(\mathrm{D}^2 u\big)^{-1}\mathrm{D}u,$$

so that $y = \big(\mathrm{D}^2 u\big)^{-1}\mathrm{D}u$ is tangent to the income expansion path determined by u, p and x. □

We now show that equation (4) holds.

Lemma 3.2. *Let u be as in Lemma 3.1, let w denote income, and let g denote the inverse demand function determined by u. For $x \in \mathbb{R}^\ell_{++}$ the maximum of $z^T \mathrm{D}_x g z$ over $z \in \mathbb{R}^L$ satisfying $z^T \mathrm{D}u = x^T \mathrm{D}u$ is given by $Z^* = \frac{w}{4}\left[\frac{x^T \mathrm{D}u}{\mathrm{D}u^T \big(\mathrm{D}^2 u\big)^{-1}\mathrm{D}u} - \frac{x^T \mathrm{D}^2 u\ x}{x^T \mathrm{D}u} - 4\right]$.*

Proof. The maximum is given by ${z^*}^T \mathrm{D}_x g z^*$ where z^* is given by equation (5). As

$$\mathrm{D}_x g y = \left[\frac{\mathrm{D}^2 u}{x^T \mathrm{D}u} - \frac{\mathrm{D}u\mathrm{D}u^{\mathrm{T}}}{\big(x^T \mathrm{D}u\big)^2} - \frac{\mathrm{D}u\big(\mathrm{D}^2 u\ x\big)^{\mathrm{T}}}{\big(x^T \mathrm{D}u\big)^2}\right]\big(\mathrm{D}^2 u\big)^{-1}\mathrm{D}u$$

$$= \frac{\mathrm{D}u}{x^T \mathrm{D}u} - \frac{p^T y}{x^T \mathrm{D}u}\mathrm{D}u - \frac{\mathrm{D}u}{x^T \mathrm{D}u} = -\frac{p^T y}{x^T \mathrm{D}u}\mathrm{D}u$$

$$\mathrm{D}_x g x = \left[\frac{\mathrm{D}^2 u}{x^T \mathrm{D}u} - \frac{\mathrm{D}u\mathrm{D}u^{\mathrm{T}}}{\big(x^T \mathrm{D}u\big)^2} - \frac{\mathrm{D}u\big(\mathrm{D}^2 u\ x\big)^{\mathrm{T}}}{\big(x^T \mathrm{D}u\big)^2}\right]x = \frac{\mathrm{D}^2 u x}{x^T \mathrm{D}u} - \frac{\mathrm{D}u}{x^T \mathrm{D}u} - \frac{x^T \mathrm{D}^2 u\ x}{\big(x^T \mathrm{D}u\big)^2}\mathrm{D}u$$

$$\mathrm{D}_x g z^* = \frac{\frac{1}{2}\mathrm{D}^2 u x}{x^T \mathrm{D}u} - \left[\frac{x^T \mathrm{D}u + \frac{1}{2}x^T \mathrm{D}^2 u\ x}{\big(x^T \mathrm{D}u\big)^2}\right]\mathrm{D}u$$

$$y^T \mathrm{D}_x g z^* = \frac{1}{2} - \left[\frac{x^T \mathrm{D}u + \frac{1}{2}x^T \mathrm{D}^2 u\ x}{x^T \mathrm{D}u}\right](p^T y) \qquad x^T \mathrm{D}_x g z^* = -1$$

$${z^*}^T \mathrm{D}_x g z^* = \frac{1}{4p^T y} - \frac{1}{2}\left[\frac{x^T \mathrm{D}u + \frac{1}{2}x^T \mathrm{D}^2 u\ x}{x^T \mathrm{D}u}\right] - \frac{1}{2} = \frac{1}{4}\left[\frac{x^T \mathrm{D}u}{\mathrm{D}u^T \big(\mathrm{D}^2 u\big)^{-1}\mathrm{D}u} - \frac{x^T \mathrm{D}^2 u\ x}{x^T \mathrm{D}u} - 4\right].$$

□

Remark 1. Note that since

$$\frac{x^T \mathrm{D}_x g x}{(p^T x)^2} = \frac{y^T \mathrm{D}_x g y}{(p^T y)^2} = -1 < 0, \tag{6}$$

the ULD can not be violated in the direction of either x or y. For a commodity change in the direction of x, $\Delta x = \theta x$ with $\theta \in (-1, \infty) \setminus \{0\}$, relative *consumption* remains constant so that:

$$\frac{x + \Delta x}{x_1 + \Delta x_1} = \frac{x}{x_1}.$$

Prices changes Δp for which $\Delta p^T x > 0$ move the budget hyperplane inward from x, $w = p^T x$, $\Delta p^T x > 0 \implies (p + \Delta p)^T x > w$, and if $\Delta p^T x < 0$, the budget hyperplane moves outward from x, $\Delta p^T x < 0 \implies (p + \Delta p)^T x < w$. Therefore if consumption changes in the direction of the original commodity bundle x, prices move in the opposite direction. This relationship is illustrated in left panel of Figure 1.

Remark 2. For an instantaneous commodity change in the direction of y, $\mathrm{d}x = \theta y$ with $\theta \in (-1, \infty) \setminus \{0\}$, relative *prices* remain constant as y is tangent to the income expansion curve. Suppose x maximizes utility given prices p and income w and x' maximizes utility given prices p' with constant income w. If $p' = rp$ with $r \in \mathbb{R}^+ \setminus \{1\}$ then:

$$w = p^T x = p'^T x' \implies (p' - p)^T (x' - x) = -\frac{(1-r)^2 w}{r} < 0.$$

A change in prices Δp which is proportional to current prices is normal to the budget hyperplane $p^T x = w$ and moves the budget hyperplane parallel in the opposite direction. Thus prices and consumption move in opposite directions. This relationship is illustrated in right panel of Figure 1.

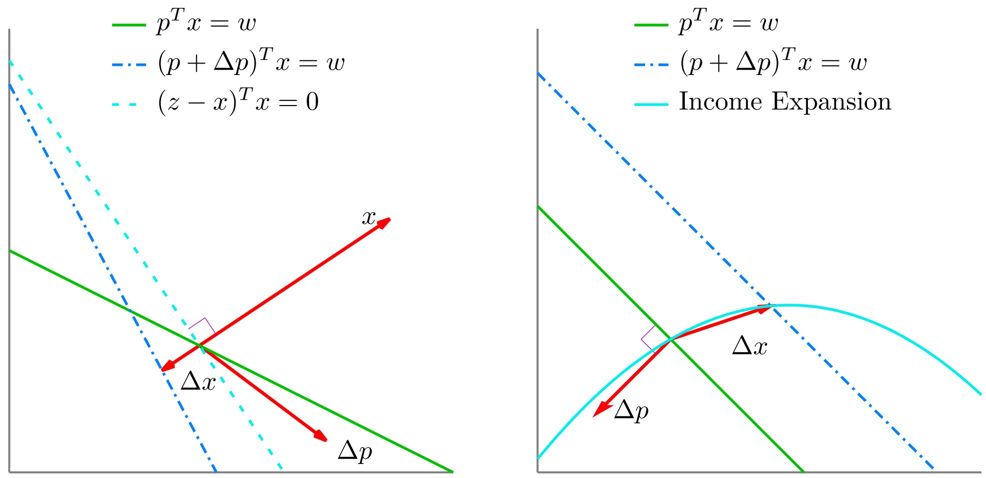

Figure 1: Left Panel: When consumption ratios remain constant, prices and consumption move in opposite directions. Right Panel: When relative prices remain constant, prices and consumption move in opposite directions.

Remark 3. Equation (5) which leads to the main result of our paper, Proposition 3.3, shows that the M-P necessary condition may be interpreted as checking the 'single direction' of a commodity demand change halfway between the tangent to the income expansion path and the original consumption point, x.

Proposition 3.3. *Let u be as in Lemma 3.1 and let g denote the inverse demand function determined by u. For each bundle x in the commodity space we have that* $\max \mathrm{d}x^T \mathrm{d}p = {z^*}^T \mathrm{D}g z^*$ *and therefore that the ULD holds if and only if* ${z^*}^T \mathrm{D}g z^* < 0$.

Proof. Follows from Lemma 3.2. □

Remark 4. This seems to be an interesting geometrical way to think about the ULD. In fact one could interpret this as a comparison between the actual preferences and a 'homothetic extension' of a single indifference surface.

Definition 2. [Homothetic extension]: A family of level sets (or indifference surfaces), is a homothetic extension of a single level set if the slope of each member of the family at a point where a ray from the origin cuts it, is the same as it is at the point where the ray cuts the given level set.

Figure 2 illustrates these ideas.

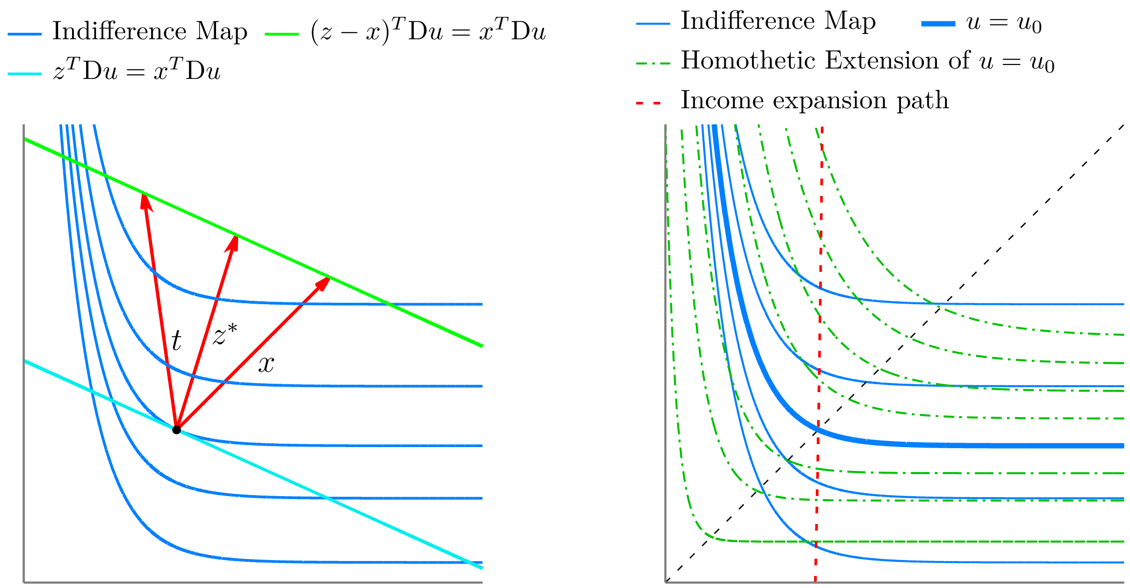

Figure 2: The left panel shows some indifference curves of u as well as vectors $t = y/(p^Ty)$, $x = (x_1, x_2)^T$ and the Giffen commodity vector $z^* = \frac{1}{2}(t + x)$. The right panel shows some indifference curves of u as well as some homothetic extension of a single indifference curve.

3.1 Example 1 the M-P sufficient condition

Mityushin and Polterovich (1978) pointed out that the assumed concavity of $u(x)$ makes the first term on the left-hand side of (2) non-positive. Therefore a sufficient condition for the monotonicity of $x(p)$ at p is that when $x = x(p)$:

$$-\frac{x^T\mathrm{D}^2ux}{x^T\mathrm{D}u} < 4 \tag{7}$$

Given its relative simplicity, the sufficient condition for the ULD in (7) is more widely encountered than the necessary and sufficient condition in (2). However, since equation (2) provides a necessary and sufficient condition for a demand function to be monotone, while equation (7) provides a merely sufficient condition, examples of utility functions which just fail to satisfy equation (2) exist as well as utility functions which fail to satisfy (7). We first consider an example of simple additive utility function. We contrast the calculation involved to establish a violation of ULD using our criterion with that involved using the M-P criteria. We argue that ours is easier to verify and that it is based somewhat more on observables.

Let ϵ denote an arbitrary positive constant and consider the utility function defined by,

$$u(x_1, x_2) = -e^{6(1+\epsilon)(1-x_1)} - e^{3\epsilon(1-x_2)} + 3x_2. \tag{8}$$

The partial derivatives of u, $\partial u/\partial x_1 = 6(1+\epsilon)e^{6(1+\epsilon)(1-x_1)}$ and $\partial u/\partial x_2 = 3 + 3\epsilon e^{3\epsilon(1-x_2)}$ are positive and the Hessian, $\mathrm{D}^2u = \begin{bmatrix} -36(1+\epsilon)^2e^{6(1+\epsilon)(1-x_1)} & 0 \\ 0 & -9\epsilon^2e^{\epsilon(1-x_2)} \end{bmatrix}$ is negative definite. With $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

$$\mathrm{D}u = \begin{pmatrix} 6(1+\epsilon) \\ 3(1+\epsilon) \end{pmatrix} \qquad x^T\mathrm{D}u = 9(1+\epsilon) \quad \mathrm{D}^2u = \begin{bmatrix} -36(1+\epsilon)^2 & 0 \\ 0 & -9\epsilon^2 \end{bmatrix}.$$

Computing the M-P sufficient condition, equation (7), gives

$$-\frac{x\mathrm{D}^2ux}{x^T\mathrm{D}u} = \frac{36(1+\epsilon)^2 + 9\epsilon^2}{9(1+\epsilon)} = 4 + 4\epsilon + \frac{\epsilon^2}{1+\epsilon} \longrightarrow 4^+ \text{ as } \epsilon \to 0^+.$$

To see that u gives rise to demands that violate the ULD, consider the inverse demand function, g. Then we have:

$$\mathrm{D}_x g = \frac{\mathrm{D}^2 u}{x^T \mathrm{D}u} - \frac{\mathrm{D}u\mathrm{D}u^{\mathrm{T}}}{\left(x^T\mathrm{D}u\right)^2} - \frac{\mathrm{D}u\left(\mathrm{D}^2 u\; x\right)^{\mathrm{T}}}{\left(x^T\mathrm{D}u\right)^2}$$

$$= \frac{\mathrm{D}^2 u}{9(1+\epsilon)} - \frac{1}{[9(1+\epsilon)]^2}\begin{pmatrix}6(1+\epsilon)\\3(1+\epsilon)\end{pmatrix}\begin{pmatrix}6(1+\epsilon) & 3(1+\epsilon)\end{pmatrix} - \frac{1}{9(1+\epsilon)}\begin{pmatrix}6(1+\epsilon)\\3(1+\epsilon)\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix}\frac{\mathrm{D}^2 u}{9(1+\epsilon)}$$

$$= \begin{bmatrix}-4(1+\epsilon) & 0\\ 0 & \frac{-\epsilon^2}{1+\epsilon}\end{bmatrix} - \frac{1}{9}\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}2 & 1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix}\begin{bmatrix}-4(1+\epsilon) & 0\\ 0 & \frac{-\epsilon^2}{1+\epsilon}\end{bmatrix}$$

$$= \begin{bmatrix}-4(1+\epsilon) & 0\\ 0 & \frac{-\epsilon^2}{1+\epsilon}\end{bmatrix} - \frac{1}{9}\begin{bmatrix}4 & 2\\ 2 & 1\end{bmatrix} - \frac{1}{3}\begin{bmatrix}2 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}-4(1+\epsilon) & 0\\ 0 & \frac{-\epsilon^2}{1+\epsilon}\end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix}1 & -2\\ -1 & 2\end{bmatrix}\begin{bmatrix}-4(1+\epsilon) & 0\\ 0 & \frac{-\epsilon^2}{1+\epsilon}\end{bmatrix} - \frac{1}{9}\begin{bmatrix}4 & 2\\ 2 & 1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}-4(1+\epsilon) & \frac{2\epsilon^2}{1+\epsilon}\\ 4(1+\epsilon) & \frac{-2\epsilon^2}{1+\epsilon}\end{bmatrix} - \frac{1}{9}\begin{bmatrix}4 & 2\\ 2 & 1\end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix}-16-12\epsilon & -\frac{2(1+\epsilon-3\epsilon^2)}{1+\epsilon}\\ 10+12\epsilon & -\frac{1+\epsilon+6\epsilon^2}{1+\epsilon}\end{bmatrix}$$

Rather than compute z^* given in equation (3) we set for simplicity, $\mathrm{d}x = \begin{pmatrix}1\\4\end{pmatrix}$. Assuming $\epsilon < 1$,

$$\mathrm{d}x^T \mathrm{D}_x g \mathrm{d}x = \frac{1}{9}\begin{pmatrix}1 & 4\end{pmatrix}\begin{bmatrix}-16-12\epsilon & -\frac{2(1+\epsilon-3\epsilon^2)}{1+\epsilon}\\ 10+12\epsilon & -\frac{1+\epsilon+6\epsilon^2}{1+\epsilon}\end{bmatrix}\begin{pmatrix}1\\4\end{pmatrix} = \frac{1}{9}\begin{pmatrix}1 & 4\end{pmatrix}\begin{pmatrix}-24-12\epsilon+\frac{24\epsilon^2}{1+\epsilon}\\ 6+12\epsilon-\frac{24\epsilon^2}{1+\epsilon}\end{pmatrix}$$

$$= \frac{1}{9}\left(36\epsilon - \frac{72\epsilon^2}{1+\epsilon}\right) = 4\epsilon - \frac{8\epsilon^2}{1+\epsilon} = \frac{4\epsilon-4\epsilon^2}{1+\epsilon} > 0,$$

so that the demand behaviour generated by u violates the ULD. It follows that u does not satisfy the M-P necessary and sufficient condition in equation (2). Indeed,

$$\frac{x^T\mathrm{D}u}{\mathrm{D}u^T(\mathrm{D}^2u)^{-1}\mathrm{D}u} - \frac{x^T\mathrm{D}^2ux}{x^T\mathrm{D}u} = 4 + \frac{4\epsilon(1+\epsilon)+\epsilon^4}{(1+\epsilon)^3+\epsilon^2(1+\epsilon)}, \tag{9}$$

as the following calculations show.

$$\mathrm{D}u^T(\mathrm{D}^2u)^{-1}\mathrm{D}u = \begin{pmatrix}6(1+\epsilon) & 3(1+\epsilon)\end{pmatrix}\begin{bmatrix}-(1+\epsilon)^{-2}/36 & 0\\ 0 & -\epsilon^{-2}/9\end{bmatrix}\begin{pmatrix}6(1+\epsilon)\\3(1+\epsilon)\end{pmatrix}$$

$$= \begin{pmatrix}6(1+\epsilon) & 3(1+\epsilon)\end{pmatrix}\begin{pmatrix}-(1+\epsilon)^{-1}/6\\ -(1+\epsilon)\epsilon^{-2}/3\end{pmatrix} = -1 - \frac{(1+\epsilon)^2}{\epsilon^2} = -\frac{(1+\epsilon)^2+\epsilon^2}{\epsilon^2}$$

$$\frac{x^T\mathrm{D}u}{\mathrm{D}u^T(\mathrm{D}^2u)^{-1}\mathrm{D}u} - \frac{x^T\mathrm{D}^2ux}{x^T\mathrm{D}u} = -\frac{9(1+\epsilon)\epsilon^2}{(1+\epsilon)^2+\epsilon^2} + 4 + 4\epsilon + \frac{\epsilon^2}{1+\epsilon} = 4 + \epsilon\left[4 + \frac{\epsilon}{1+\epsilon} - \frac{9\epsilon(1+\epsilon)}{(1+\epsilon)^2+\epsilon^2}\right]$$

$$= 4 + \epsilon\left[\frac{\epsilon}{1+\epsilon} + \frac{4(1+\epsilon)^2 - 9\epsilon(1+\epsilon)+4\epsilon^2}{(1+\epsilon)^2+\epsilon^2}\right]$$

$$= 4 + \frac{\epsilon^2}{1+\epsilon} + \frac{4\epsilon-\epsilon^2(1+\epsilon)}{(1+\epsilon)^2+\epsilon^2} = 4 + \frac{4\epsilon(1+\epsilon)+\epsilon^4}{(1+\epsilon)^3+\epsilon^2(1+\epsilon)} \longrightarrow 4^+ \text{ as } \epsilon \to 0^+.$$

Utilizing the above calculations we determine z^* as given in equation (3).

$$z^* = \frac{1}{2}x + \frac{\frac{1}{2}x^T \mathrm{D}u}{\mathrm{D}u^T \left(\mathrm{D}^2 u\right)^{-1} \mathrm{D}u} \left(\mathrm{D}^2 u\right)^{-1} \mathrm{D}u = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} - \frac{\frac{9}{2}(1+\epsilon)\epsilon^2}{(1+\epsilon)^2+\epsilon^2}\begin{pmatrix}-(1+\epsilon)^{-1}/6\\-(1+\epsilon)\epsilon^{-2}/3\end{pmatrix}$$
$$= \begin{pmatrix}0.5\\0.5\end{pmatrix} + \frac{3}{4}\left[(1+\epsilon)^2+\epsilon^2\right]^{-1}\begin{pmatrix}\epsilon^2\\2(1+\epsilon)\end{pmatrix}$$

3.1.1 Numerical Illustration

For a numerical illustration let $\epsilon = 1/3$ so that $u = -e^{8(1-x_1)} - e^{1-x_2} + 3x_2$. For $x = \begin{pmatrix}1\\1\end{pmatrix}$,

$$\mathrm{D}u = \begin{pmatrix}8\\4\end{pmatrix}, \quad x^T \mathrm{D}u = 12 \quad \text{and} \quad \mathrm{D}^2 u = \begin{bmatrix}-64 & 0\\0 & -1\end{bmatrix}.$$

Computing the M-P sufficient condition,

$$-\frac{x^T \mathrm{D}^2 u x}{x^T \mathrm{D}u} = \frac{65}{12} = 5.42 > 4.$$

Checking the M-P necessary and sufficient condition,

$$\left(\mathrm{D}^2\right)^{-1} u = \begin{bmatrix}-1/64 & 0\\0 & -1\end{bmatrix} \qquad \mathrm{D}u^T (\mathrm{D}^2 u)^{-1} \mathrm{D}u = -17,$$

so that

$$\frac{x^T \mathrm{D}u}{\mathrm{D}u^T (\mathrm{D}^2 u)^{-1} \mathrm{D}u} - \frac{x^T \mathrm{D}^2 u x}{x^T \mathrm{D}u} = -\frac{12}{17} + \frac{65}{12} = 4.71 > 4.$$

If as before, $\mathrm{d}x = \begin{pmatrix}1\\4\end{pmatrix}$, $\mathrm{d}x^T \mathrm{D}_x g \mathrm{d}x = 2/3$. Note, $z^* = \begin{pmatrix}0.544\\1.912\end{pmatrix}$ so that $\begin{pmatrix}1\\4\end{pmatrix} \approx 2z^*$. Figure 3 illustrates this example with $\mathrm{d}x = 0.2z^*$.

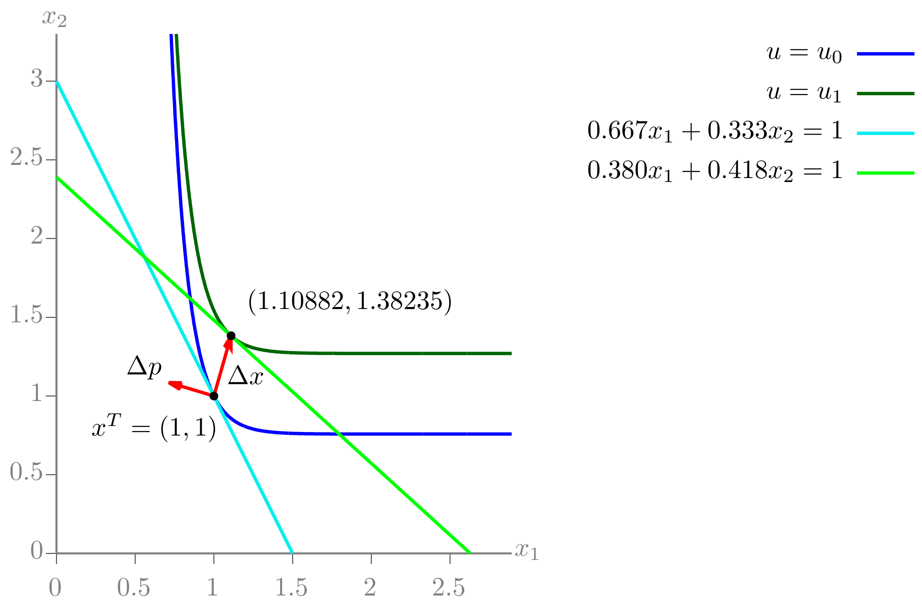

Figure 3: As the price of good x_2 rises from 0.33 to 0.42, consumption *increases* form 1 to 1.3824. The vectors shown are $\Delta x = 0.2z^* = \begin{pmatrix}0.109 & 0.382\end{pmatrix}^T$ and $\Delta p = \begin{pmatrix}-0.286 & 0.085\end{pmatrix}^T$.

3.2 Example 2 the M-P necessary and sufficient condition

Equation (2) provides a necessary and sufficient condition for the ULD while equation (7) provides a merely sufficient condition. We now give an example of a utility function which satisfies equation (2) but fails to satisfy equation (7). Let ϵ denote an arbitrary positive constant and consider a slightly different utility function defined by,

$$u(x_1, x_2) = -e^{4(1+\epsilon)(1-x_1)} - e^{\epsilon(1-x_2)}. \tag{10}$$

The partial derivatives of u, $\partial u/\partial x_1 = 4(1+\epsilon)e^{4(1+\epsilon)(1-x_1)}$ and $\partial u/\partial x_2 = \epsilon e^{\epsilon(1-x_2)}$ are positive and the Hessian, $\mathrm{D}^2 u = \begin{bmatrix} -16(1+\epsilon)^2 e^{4(1+\epsilon)(1-x_1)} & 0 \\ 0 & -\epsilon^2 e^{\epsilon(1-x_2)} \end{bmatrix}$ is negative definite. With $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

$$\mathrm{D}u = \begin{pmatrix} 4(1+\epsilon) \\ \epsilon \end{pmatrix} \qquad x^T \mathrm{D}u = 4 + 5\epsilon \quad \mathrm{D}^2 u = \begin{bmatrix} -16(1+\epsilon)^2 & 0 \\ 0 & -\epsilon^2 \end{bmatrix}.$$

Computing the M-P sufficient condition, equation (7), gives

$$-\frac{x^T \mathrm{D}^2 u x}{x^T \mathrm{D}u} = \frac{16(1+\epsilon)^2 + \epsilon^2}{4+5\epsilon} = 4 + \frac{12\epsilon + 17\epsilon^2}{4+5\epsilon} = 4 + 3\epsilon + \frac{2\epsilon^2}{4+5\epsilon} \longrightarrow 4^+ \text{ as } \epsilon \to 0^+.$$

Considering the M-P necessary and sufficient condition,

$$\begin{aligned} \mathrm{D}u^T (\mathrm{D}^2 u)^{-1} \mathrm{D}u &= \begin{pmatrix} 4(1+\epsilon) & \epsilon \end{pmatrix} \begin{bmatrix} -(1+\epsilon)^{-2}/16 & 0 \\ 0 & -\epsilon^{-2} \end{bmatrix} \begin{pmatrix} 4(1+\epsilon) \\ \epsilon \end{pmatrix} \\ &= \begin{pmatrix} 4(1+\epsilon) & \epsilon \end{pmatrix} \begin{pmatrix} -(1+\epsilon)^{-1}/4 \\ -\epsilon^{-1} \end{pmatrix} = -2 \end{aligned}$$

so that

$$\frac{x^T \mathrm{D}u}{\mathrm{D}u^T (\mathrm{D}^2 u)^{-1} \mathrm{D}u} - \frac{x^T \mathrm{D}^2 u x}{x^T \mathrm{D}u} = -2 - 2.5\epsilon + 4 + 3\epsilon + \frac{2\epsilon^2}{4+5\epsilon} = 2 + 0.5\epsilon + \frac{2\epsilon^2}{4+5\epsilon} < 4,$$

provided $\epsilon < \frac{4}{9}(2 + \sqrt{13})$[5]. Thus although u fails to satisfy equation (7) u does satisfy equation (2) and therefore determines demands which satisfy the ULD.

Utilizing the above calculations we determine z^* as given in equation (3).

$$\begin{aligned} z^* &= \frac{1}{2} x + \frac{\frac{1}{2} x^T \mathrm{D}u}{\mathrm{D}u^T \left(\mathrm{D}^2 u\right)^{-1} \mathrm{D}u} \left(\mathrm{D}^2 u\right)^{-1} \mathrm{D}u = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{4}(4+5\epsilon) \begin{pmatrix} \frac{1}{4}(1+\epsilon)^{-1} \\ \epsilon^{-1} \end{pmatrix} \\ &= \frac{1}{4} \left[\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 + \epsilon/(4+4\epsilon) \\ 5 + 4/\epsilon \end{pmatrix} \right] = \frac{1}{4} \begin{pmatrix} 3 + \epsilon/(4+4\epsilon) \\ 7 + 4/\epsilon \end{pmatrix} \end{aligned}$$

Note that $\mathrm{D}^2 u = -\begin{bmatrix} 4(1+\epsilon) & 0 \\ 0 & \epsilon \end{bmatrix}^2$ and $\mathrm{D}u = \begin{bmatrix} 4(1+\epsilon) & 0 \\ 0 & \epsilon \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ so that,

$$\begin{aligned} \mathrm{D}_x g &= \frac{\mathrm{D}^2 u}{x^T \mathrm{D}u} - \frac{\mathrm{D}u \mathrm{D}u^{\mathrm{T}}}{\left(x^T \mathrm{D}u\right)^2} - \frac{\mathrm{D}u \left(\mathrm{D}^2 u \; x\right)^{\mathrm{T}}}{\left(x^T \mathrm{D}u\right)^2} \\ &= \frac{\mathrm{D}^2 u}{4+5\epsilon} - \frac{1}{(4+5\epsilon)^2} \begin{pmatrix} 4(1+\epsilon) \\ \epsilon \end{pmatrix} \begin{pmatrix} 4(1+\epsilon) & \epsilon \end{pmatrix} - \frac{1}{(4+5\epsilon)^2} \begin{pmatrix} 4(1+\epsilon) \\ \epsilon \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \mathrm{D}^2 u \\ &= -\frac{1}{(4+5\epsilon)^2} \begin{bmatrix} 4(1+\epsilon) & 0 \\ 0 & \epsilon \end{bmatrix} \left\{ \begin{bmatrix} 4+5\epsilon & 0 \\ 0 & 4+5\epsilon \end{bmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{bmatrix} -3-4\epsilon & 0 \\ 0 & 1-\epsilon \end{bmatrix} \right\} \begin{bmatrix} 4(1+\epsilon) & 0 \\ 0 & \epsilon \end{bmatrix} \\ &= -\frac{1}{(4+5\epsilon)^2} \begin{bmatrix} 4(1+\epsilon) & 0 \\ 0 & \epsilon \end{bmatrix} \begin{bmatrix} 1+\epsilon & 1-\epsilon \\ -3-4\epsilon & 5+4\epsilon \end{bmatrix} \begin{bmatrix} 4(1+\epsilon) & 0 \\ 0 & \epsilon \end{bmatrix}. \end{aligned}$$

[5] For $\epsilon > 0$, $2 + 0.5\epsilon + \frac{2\epsilon^2}{4+5\epsilon} < 4 \iff 9\epsilon^2 - 16\epsilon - 16 < 0 \iff \epsilon < \frac{8+4\sqrt{13}}{9}$

4 Conclusion

The well known Slutsky equation characterizes the monotonicity of demand in terms of 'substitution' and 'income' effects. At some point in many expositions of Slutsky it is remarked that 'provided the income effect doesn't outweigh the substitution effect, then consumer demand will behave conventionally' – i.e. the ULD will hold (see the sources cited in Footnote 1.) It then occurs to ask: Exactly when is it that 'the income effect doesn't outweigh the substitution effect'? M-M and M-P provided sufficient and M-P provided necessary and sufficient conditions for this to be the case and for the ULD to hold. By observing that $(\mathrm{D}^2 u)^{-1}\mathrm{D}u$ is tangent to the income expansion path, we have come up with an interesting (geometrical) way of thinking about the ULD. This leads to a criterion for the ULD to hold which is both: (i) necessary and sufficient; (ii) simpler to compute and apply than is the M-P necessary and sufficient condition; and (iii) has a 'Revealed Preference' character in that it is built out of observables.

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