Macroeconomics and Inequality: The US Experience with 'r-g' and Wealth Inequality, and the role of Income Risk.

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Abstract

US experience suggests a negative or zero correlation between wealth inequality and r-g (the difference between the average rate of return on wealth and the growth rate of real income). Looking more closely at the individual factors over the past few decades the US has experienced: (i) a negative correlation between the growth rate of real income and the ratio of national wealth to income, with the growth rate of real income decreasing and the ratio of national wealth to income increasing; (ii) a negative correlation between wealth inequality and r, with wealth inequality increasing and r decreasing; (iii) a positive correlation between income inequality and wealth inequality, with income inequality increasing; and (iv) a positive correlation between income inequality and income risk, with income risk increasing. We show analytically that all four correlations can been understood jointly as arising in a neoclassical growth model with heterogeneous agents, incomplete markets and borrowing constraints. An appendix demonstrates robustness of these correlations to the use of other data sources, countries and time periods. We conclude that the model is a good starting point for jointly understanding some of the major trends of the recent decades in the US economy: trends both Macroeconomic and in Inequality. We conclude that conceptually the relation between wealth inequality and r-g is better understood as the combination of two different relationships with r and g individually. We further conclude empirically that r-g is not an important driver of wealth inequality, neither in recent US history, nor more generally.

Keywords: Wealth Inequality, Income Inequality, Rate of Return to Wealth, Economic Growth, Income Risk, Heterogeneous agents.

JEL Classification:

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Contents

This document is in progress. The data work in the body of the paper, and the raw analytical results are fine. Needing work are the analytical results which are raw and need to be reorganised to fit the structure of the rest of the paper. The introduction and most of the body of the paper read passably.

1 Introduction

What is the relationship between r-g, the difference between the rate of return to wealth and the rate of income growth, and wealth inequality? In *Capital in the Twenty-First Century* Thomas Piketty describes three laws-of-capitalism, the third of which is that r-g is positively related to wealth inequality. We show that the recent US experience is in fact the opposite: a negative or zero correlation between r-g (which has likely decreased) and wealth inequality (which has increased); 'likely' as it depends on the measurement of r, of which more later.

This negative or zero correlation between likely decreasing r-g and increasing wealth inequality captures some of the major trends in the US economy since the mid-1980s. On the grounds that combining r and g may be obscuring their importances and to better capture the outlines of the US experience we document a few other trends and emphasise a few other correlations. Alongside rising wealth inequality has been a rise in income inequality, giving us a positive correlation between income inequality and wealth inequality.[1] We then look closer at r-g, the difference between the rate of return to wealth and the rate of income growth. We see that the growth rate of output, g, has itself has fallen only that this has been (likely) outweighed by the size of the fall in the rate of return to wealth. We document a rise in the wealth-income ratio and a negative correlation between the wealth-income ratio and the rate of income growth. Lastly, the importance of rising income inequality is undoubtedly related to the form it takes, we therefore document a rise in income risk and a positive correlation between income risk and income inequality.

This leaves us summarizing recent US experience with four correlations: (i) a negative correlation between the rate of income growth and the wealth-income ratio; (ii) a negative correlation between wealth inequality and the rate of return to wealth;[2] (iii) a positive correlation between income inequality and wealth inequality; and (iv) a positive correlation between income risk and income inequality.

We show analytically that all four correlations arise jointly in the neoclassical growth model

[1] The reader may be forgiven for greeting the finding of a positive correlation between income and wealth inequality with a yawn and perhaps a derisory 'Well duh!', something your author would be liable to do. It is included on the grounds that it constitutes a basic fact of inequality that any model of inequality must be capable of displaying it.

[2] Note that this differs slightly for the wording of (ii) in the abstract. In the empirical results we show both, but this wording is closer to what we will show for the model.

with heterogeneous agents and incomplete markets.[3];[4]

This illustration of four correlations relating major trends in Macroeconomic outcomes and inequality the the US experience since the mid-1980s, together with the analytical finding that these same four correlations can be made sense of using the neoclassical growth model with heterogeneous agents and incomplete markets, is the main result and contribution of this paper. We interpret this as illustrating the importance of jointly studying macroeconomics and inequality, and as displaying the usefulness of heterogeneous agent models to do so.

The remainder of the paper includes a discussion of the choice of data sources underlying the empirical findings of these four correlations. Appendix A provides an in-depth look at robustness to other data sources, measurement concepts, time periods and countries; and we provide a short summary discussion of this as part of the paper itself. We also further discuss the relationship between r-g and Wealth inequality, and why the model predicts a negative correlation between r-g and wealth inequality, as seen not just in the US but more often than not in other countries and time periods, and why this differs from the models underlying Piketty (2014)'s 'third law-of-capitalism' which argued that the correlation should be positive.

In the remainder of the introduction we first describe how the theory of wealth inequality presented in the neoclassical growth model with heterogeneous agents and incomplete markets, with it's prediction of a negative correlation between r-g and wealth inequality, differs from the models favoured by Piketty (2014) with their prediction of a positive correlation. We then turn to a lengthy but important discussion of the choice of data sources used to illustrate the trends and correlations seen in the US over recent decades; since we wish to use a model to jointly understand the relationships between trends in the macroeconomy and inequality emphasis is given on trying to use micro and macro measures that fit together. We then provide a short discussion of some features the neoclassical growth model with heterogeneous agents and incomplete markets lacks, but which provide important context when interpreting our findings in relation to inequality.

Section 2 provides a simple illustration of some major trends in the US economy since the mid-1980s, both in terms of Macroeconomic outcomes and inequality. We emphasise four correlations: (i) a negative correlation between the rate of income growth and the wealth-income ratio; (ii) a negative correlation between wealth inequality and the rate of return to wealth. (iii) a positive correlation between income inequality and wealth inequality; and (iv) a positive correlation between income risk and income inequality.

Section 3 looks at the neoclassical growth model with heterogeneous agents and incomplete markets. We introduce a definition for a *balanced growth path*. We show analytically that the

[3] In the model we show a negative correlation between r and wealth in equality, for a given g. This is not exactly what we showed in the data, nor does it precisely pertain to Piketty's third law-of-capitalism which is about r-g and wealth inequality.

[4] In Appendix D we provide the extension of many of the analytical results presented for the neoclassical growth model with heterogeneous agents and incomplete markets to include endogenous labour supply decisions.

model reproduces the four correlations. Appendix D shows that these results all continue to hold in the extension of the model to endogenous labour supply.

Appendix A shows that the trends and correlations we highlight for the US in recent decades are robust to alternative measures of many of the concepts, such as r and wealth inequality. We also show that the correlations are in fact typical of other time periods and countries, especially over recent decades. (For many countries recent decades are the only ones where sufficient data is available to look at this relationship.) We find that *contrary* to Piketty's third law-of-capitalism the correlation between r-g and wealth inequality is more often negative than not, especially over recent decades. For a few cases the positive correlation predicted by Piketty is observed, but these are very much the exception rather than the rule. Our analytical results provide an easy to understand explanation: high levels of wealth inequality are associated with high ratios of wealth to income and hence to low levels of r.[5] This effect is large enough to overpower the increased incentives to save that might come from an increase in r; that is, the mechanism posited by Piketty is present in the model, but overwhelmed by forces pushing in the opposite direction.

Relation to other theories of Wealth Inequality: Piketty (2014)'s third law-of-capitalism — a positive correlation between r-g and wealth inequality — was not summoned out of thin air. It is well founded in a specific family of models in wealth inequality. In these models a higher r-g implies an increase in the steady-state inequality. The model combines a balanced-path-growth model with a Pareto distribution of wealth. An increase in r-g leads to increasing wealth inequality because r-g determines the rate at which assets will accumulate over time. A higher r-g allows individuals with higher assets to have higher incomes, and since savings are a fixed fraction of income, or at least not declining, leading to ever higher assets relative to other individuals. Jones (2015) provides a broad overview of these models; and a more in-depth technical treatment of the models themselves can be found in the Technical Appendix thereof. The Appendix to Chapter 10 of Piketty (2014) refers to these models as those of which he is thinking when arguing that there is a positive relationship between r-g and inequality. For the readers convenience Appendix E provides a brief example of a model of this type.

So why does the neoclassical growth model with heterogeneous agents and incomplete markets predict the opposite: a negative correlation between r-g and wealth inequality? The key differences between this model and those favoured by Piketty (2014) are the drivers of income inequality and savings decisions. In the models favoured by Piketty savings is exogenous and wealth inequality simply reflects the 'accumulation' of income inequality. In the models presented here savings is related to risk aversion and income inequality is related to income risk, so increased wealth inequality arises alongside increases in wealth more generally (an increase in the wealth-to-income ratio) and hence to decreases in the return to wealth. Income risk thus presents itself as one

[5] More precisely, high levels of income risk lead to more precautionary savings, especially among the wealthy, and this leads to higher wealth inequality and higher wealth-income ratios, which in turn leads to lower levels of r.

of the major factors differentiating the neoclassical growth model with heterogeneous agents and incomplete markets from the kinds of models underlying Piketty's third law-of-capitalism. Hence our focus on trends in income risk as one of the major trends of interest in recent US experience.

Looking at just the cross-section the neoclassical growth model with heterogeneous agents and incomplete markets becomes a standard Bewley-Huggett-Aiyagari model. We here use a parsimonious model for the cross-section to allow us to derive analytical results and to focus attention on a few major trends and correlations. This same parsimony means that the model presented here would not be able to reproduce the degree of inequality observed in the US; for reasonable parametrizations the cross-sectional model has Gini coefficients of income inequality and wealth inequality far below the US (Aiyagari, 1994).

More advanced Bewley-Huggett-Aiyagari models can quantitatively capture the degree of US income and wealth inequality, and also the (low amount of) correlation between the two, as shown by (Castaneda et al., 2003).[6] Their model endogenizes labour supply but does not fit with the extension of our main results to allow for this in Appendix D as their utility function is seperable in consumption and leisure. This seperability was done largely to allow the model to match the facts on cross-sectional hours worked.[7] Other ways to capture the facts on cross-sectional hours without needing utility functions that are seperable in consumption and leisure have since appeared all based on various mechanisms for how current hours worked affect future income (one example would be human capital accumulation; Mustre-del Rio (2015); Huggett et al. (2006)). A recent summary of the performance of Bewley-Huggett-Aiyagari models (and others) in quantitatively capturing and explaining inequality in income and wealth is given by De Nardi (2015). Thus, while we choose to focus here on a parsimonious version of the model in terms of the cross-section to allow for deriving analytical results and to focus on the main trends, extensions of the cross-sectional aspects of the neoclassical growth model with heterogeneous agents and incomplete markets are capable of quantitatively capturing and explaining many details of US inequality.

Data and Measurement: We focus the body of our paper on presenting simple results on major trends and correlations in the US economy between macroeconomic variables and inequality over the past few decades. To provide clear illustrations of these we make deliberate choices about which measures are most appropriate and present these. We now discuss why we made these choices, and talk about possible alternative measures. Appendix A shows that our results are robust to almost all of the alternative measures, even though we consider some of these alternative measures as inappropriate for our present purposes for reasons discussed below; the appendix also shows that the results tend to be either robust or insignificant in other time periods and countries.

The measurement of income growth and income inequality, while not without methodological

[6]Castaneda et al. (2003) is based on measures of inequality at the household level like those we emphasise in the present paper. See 'Data and Measurement' for more on this.

[7]Personal communication with one of the authors.

and conceptual issues, are much more advanced than the measurement of wealth inequality, the rate of return to wealth, and income risk. Appendix A does still present numerous robustness checks to alternative measures of income growth and income inequality. Here we discuss our choices of how to measure the other three: wealth inequality, the rate of return to wealth, and income risk.

Wealth inequality: We choose to measure wealth inequality for the US directly using the numbers from the Survey of Consumer Finances (SCF) for the Shares of Total Wealth held by the Top 10% Wealthiest Households, and also the Gini coefficient of Wealth. When looking at other countries (in Appendix A) we try to focus on similar surveys, or use numbers from LIS (formerly Luxembourg Income Study), we also look at measures like the 90/50 ratio — the ratio of wealth of the 90th percentile to the wealth of the 50th percentile — and the 80/20 and 90/10 ratios, and at the Inverted Pareto Coefficient of the top tail of the wealth distribution. Alternative measures do exist and we look at robustness to many of them in Appendix A. They are typically based on accruing either capital income from income tax-returns (eg., Saez and Zucman (2016) for US) or on accruing observed wealth from estate tax-returns (eg., Kopczuk and Saez (2004)). One weakness of our choice of the SCF is that while designed to oversample the wealthy/high-income it deliberately omits those on the Forbes 400 rich list, and so likely still undersamples the Top 0.1% of Wealth (Saez and Zucman, 2016); we therefore simply ignore the Top 0.1% in our main results and only consider them in Appendix A when using alternative data sources. Using the SCF does limit our analysis to the period since the mid-1980s (prior to which the SCF simply did not exist in it's modern form), but this seems a limitation worth accepting in comparison to the issues in alternative measures described below.

Measures of wealth inequality based on accruing capital income from income tax-returns suffer from one major weakness for our present purposes: they are measured at the level of tax-units. Appendix A show that our results are all robust to using the wealth inequality data of Saez and Zucman (2016). We prefer in the body of the paper however to focus on household level data. Using infinitely-lived household models is typically justified on the grounds that the model agents represent household dynasties, and so the use of household level inequality data from the SCF provides us with model-consistent results. Importantly for our purposes the Saez and Zucman (2016) methods capitalize the capital income in such a way that the capitalization factor can vary from year to year (otherwise they would implicitly be assuming a constant r).

Measures of wealth inequality based on estate tax-returns suffer from a number of lesser problems, mostly relating to issues of bias. Firstly, most of the deceased do not have to file estate tax-returns (as their estate is not large enough to necessitate it), and so the coverage of estate tax-returns is far from complete. Secondly, they do not account for the increasing difference in mortality rates between higher and lower incomes observed in the US. (Rich people are living ever longer than lower income people.) Third, while they show the same trends as other measures of wealth inequality over most of the observable time periods, they have diverged from other measures of wealth inequality in the last decade and it seems likely that this is due to weaknesses in the estate tax-return measures, rather than in other measures (Kopczuk, 2015).

Since we are using the Survey of Consumer Finances (SCF) as our source for wealth inequality it seems most appropriate to use the same source for our data on income inequality; again we focus on shares of top percentiles and the Gini coefficient. One of the nice aspects of this choice is that it is more holistic, as income inequality and wealth inequality are undoubtedly related, and the dataset also allows consideration of issues such as the relationship between who the income-rich and wealth-rich are; although we do not look at this later issue here it has been addressed elsewhere in the quantitative macroeconomics literature on heterogeneous agent models Castaneda et al. (2003). (They are largely different people.) It also has the big advantage that it means we are using the exact same definition of households to measure inequality. In Appendix A we consider robustness to the data on income inequality in the World Wealth and Income Database (formely World Top Incomes Database; and source used by Piketty (2014)); this data is measured at the level of tax-units, not households, meaning it would be quite different to our source for wealth inequality data.[8] The appendix also considers robustness to income inequality numbers from the LIS.[9] Our main source of data on the Wealth Inequality numbers from the SCF is Wolff (2010), and hence our focus on income inequality, when from the model perspective earnings inequality would be more appropriate (Wolff (2010) does not report earnings numbers). Appendix A includes a comparison of income, earnings and wealth numbers from the SCF for the few years they are available (Díaz-Giménez et al., 1997; Burdia Rodriguez et al., 2002; Díaz-Giménez et al., 2011).

Rate of Return to Wealth: We choose to measure the rate of return to wealth using the interest rate on 90-day Treasury bills. We justify this choice purely on the grounds that it has been the standard measure in existing looks at r-g (Acemoglu and Robinson, 2015), and in the literature on 'secular stagnation' which focuses on the decline in r (Summers, 2014; Sec, 2014).

As Gomme et al. (2015) observe in relation to the secular stagnation debate it is not clear that the interest rate on 90-day Treasury bills is the appropriate measure; a more direct measure of the rate of return to capital such as that constructed using capital income and a measure of the capital stock in Gomme et al. (2011) seems more appropriate.[10] The distinction between wealth and capital is an important and troublesome issue in relation to the our present focus on inequality in wealth, raising a number of difficult measurement issues (Blume and Durlauf, 2015). While our

[8]Not just in theory, also in practice. Saez and Zucman (2016) compare their new measure of wealth inequality at the tax-unit level with existing household level measures with differences on the order of ten percentage points when measuring the Share of Total Wealth held by the Top 1% of Wealth. The use of tax-units is also problematic for cross-country comparisons as the definition of tax-units differs from country to country; eg. in some countries married couples file joint tax returns, while in other countries they file seperate tax returns.

[9]Standard panel-data sets like the Panel Study of Income Dynamics (PSID) typically top-code income data — ie. there is just an 'incomes over USD100,000' category so we cannot observe actual incomes for high income households — and so are not suitable for our present purposes of looking at wealth and income inequality.

[10]Interest rates on 90-day Treasury bills will also reflect other issues such as demand for liquidity, flight to safety, perceptions of sovereign risk, and current monetary policy.

model makes no distinction between rates of return to capital and wealth (or even interest rates on government bonds for that matter) we thus feel that a more direct measure of the return to wealth would be appropriate. Appendix A therefore demonstrates robustness to a measure for the rate of return on wealth created in a similar fashion to that of Gomme et al. (2011) for capital: we divide the capital share of income (which according to our neoclassical growth model measures rK/Y) by the wealth-output ratio (K/Y) to get a measure of r, the rate of return to wealth. We use two measures of the capital share of income, those of Karabarbounis and Neiman (2014) and the OECD.[11] We measure the wealth-output ratio as the ratio of Net National Wealth to National Income, with both series taken from the World Wealth and Income Database (formely World Top Incomes Database).[12] We find that our results are robust to using this measure of the rate of return to wealth. We note however that they would not be robust to using the rate of return to capital of Gomme et al. (2011); the difference appears to be due to their finding a smaller increase in the capital-output ratio than we find for the wealth-output ratio, which in turn is related to the falling relative price of investment goods.[13]

Income Risk: We choose to measure income risk as the ratio of the variance of income to the variance of consumption. This measure is based on the life-cycle consumption/permanent income hypothesis. If changes in income are more transitive changes, like those in the model, then consumption should be smoothed in response and so we will see the variance of income increase while the variance of consumption changes little, leading to an increase in their ratio; ie. to an increase in our measure of income risk. By contrast if increases in income inequality represented changes in permanent income we would expect to see the variance of consumption go up by a similar amount to the increase in the variance of incomes; ie. no increase in our measure of income risk. A major motivation for our selection of this measure is purely practical: it is transparent and can be easily calculated from existing data sources for a variety of countries and time periods.

Our measurement of trends in income risk in the US since the mid-1980s using the ratio of the variance of income to the variance of consumption appears to fit with other findings on income risk. The trends we observe are roughly the same as found by regression-based measures that use regression techniques to decompose changes in income into a permanent (unit root) shock component, and a transitory (uncorrelated) shock component (Krueger et al., 2010; Moffitt and Gottschalk, 2009, 2012); although structural models suggest that the reduced-form regressions used in such decompositions may be misspecified Altonji et al. (2014).[14],[15]

[11]The OECD data is also presented in Karabarbounis and Neiman (2014) as part of their study.

[12]The wealth and income numbers are therefore both as measured by standard National Accounts, but with minor adjustments to the wealth numbers to correct for the ways different countries have implemented international standards Piketty and Zucman (2014).

[13]Gomme et al. (2015) show that according to their measure there has been no notable fall in r. This means no secular stagnation and overturns our findings about the negative correlation between r-g and wealth inequality, and the negative correlation between r and K/Y.

[14]An additional difficulty with such measures is seperating cohort-effects from time-period-effects, and that results often differ based on whether regression is run on level of (log) wages or first differences.

[15]These findings of rising income risk are not beyond all doubt. Sabelhaus and Song (2010) use data a one-percent

Our measure of income risk implicitly assumes that insurance against transitory shocks is only partial, and that the amount of insurance available has remained roughly constant over time. Seperating risk and insurance is a difficult issue and one we will not attempt here, see Guvenen and Smith (2014) for more.

Missing Pieces: Obviously no model of the economy is going to capture everything. But what are some of the major issues relating to recent US experiences of economic growth and inequality that the model fails to capture? We now briefly describe what we feel are two of the main issues the model misses.

One is the changing share of total income going to capital (the US has experienced an increase in the 'capital share of income' over recent decades). Appendix A documents both of these trends and shows that there is a positive correlation between the capital share of income and wealth inequality, as well as a positive correlation between the capital share of income and the capital-output ratio. While both of these two correlations might be 'understood' if we were to replace the Cobb-Douglas production function of our representative firm with a constant-elasticity-of-substitution (CES) production function this would break the general equilibrium properties of our model. We thus leave a fuller study of these aspects to future work, prefering to present here the parsimony of the neoclassical growth model with heterogeneous agents and incomplete markets. (If a model of this complexity might still be called parsimonious.)

The other is globalization. Many of the trends we observe in recent decades for the US are not unique to the US. This is particularly true of increasing inequality in income and wealth, which is widely observed in high-income countries.[16] Autor et al. (2015, 2016) show that, at least for understanding changes in the incomes of lower-income workers globalization, or more precisely import competition from China, likely plays an important role. Karabarbounis and Neiman (2014) show that the changes in the capital share of income are widespread, evident not just in high-income countries but also in countries like China and India. This suggests the changes in the capital share of income are more likely of a technological/institutional/relative-negotiating-power nature than being also due to globalization (ie. these two omissions of capital-share-of-income and globalization are likely to be two genuinely seperate omissions). The omission of globalization means that some explanations for low interest rates, such as excess global savings originating in Asia (Bernanke, 2005), are also absent from the model.

Our present analysis ignores both these issues of the capital-share-of-income and globalization. We mention them here as they provide the reader with important context for the interpretation of

sample of the Social Security Administration and find that the volatility of income has decreased. Given that all the publicly available datasets, whose weaknesses and biases are better understood and studied, point to rising volatility, and only the poorly understood and little studied closed-access datasets point to no rise in volatility it seems clear that the burden of proof remains on the closed-access datasets to demonstrate why they might be the more reliable results.

[16] Like almost any observation in economics there are exceptions, eg. prior to the crisis income inequality was falling in Spain Pijoan-Mas and Sanchez-Marcos (2010). (Inequality in Spain then rises during the crisis.)

our own findings.

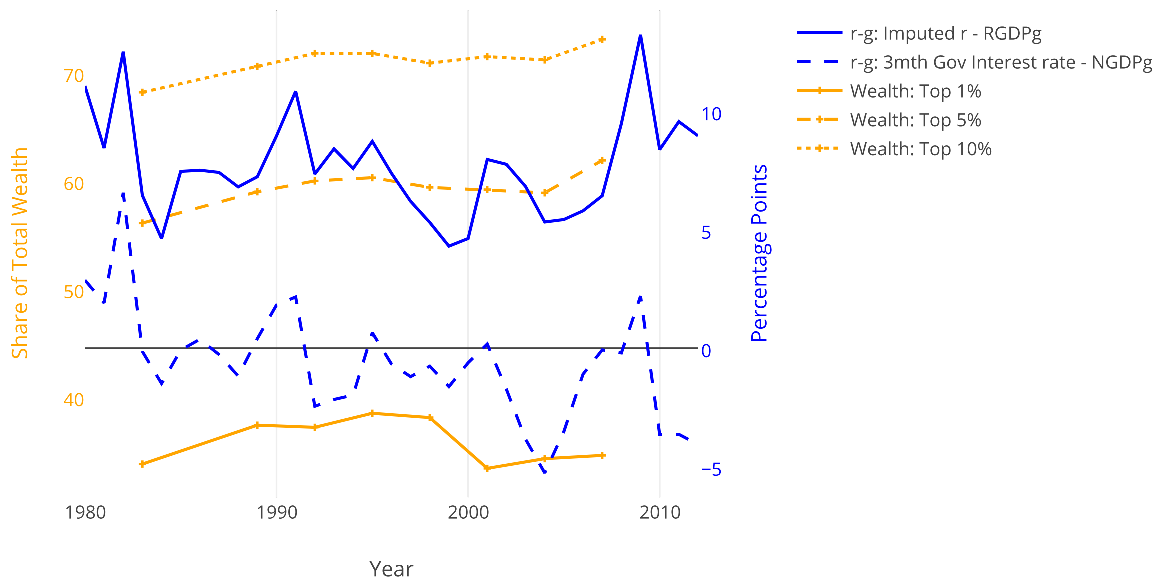
2 The US Experience since mid-1980s

We start by looking at the correlation between wealth inequality and r-g; more controversial as it relates to Piketty's third law-of-capitalism. We then turn to graphs for each of the four correlations that broadly capture the US experience in terms of macroeconomic outcomes and inequality since the mid-1980s: (i) a negative correlation between the rate of income growth and the wealth-income ratio; (ii) a negative correlation between wealth inequality and the rate of return to wealth. (iii) a positive correlation between income inequality and wealth inequality; and (iv) a positive correlation between income risk and income inequality. These four correlations are also those we later prove are predictions of the neoclassical growth model with heterogeneous agents and incomplete markets.

We feel that since the graphs presented here are quite simple it is better to allow them to speak for themselves, rather than attempt to summarize each graph as a single correlation number. Appendix A presents correlations for those who would like a single number. The same Appendix also covers a wide range of other data sources (including IMF, OECD, LIS, WWID, SCF, PWT8 and various papers), time periods, and countries; we will make some summary reference to those findings in this section. The choice of mid-1980s as starting date is purely pragmatic: it is the first period from which household level data on income and wealth inequality is available for the top percentiles. Any reader troubled by our choice of mid-1980s start date, of data source, of measurement concept, of graphs rather than regression output, or even our focus on the US will find Appendix A worthwhile.

Negative or Zero Correlation between Wealth Inequality and r-g:

Figure 1 shows the negative or zero correlation between wealth inequality and r-g — the difference between the rate of return to wealth and the rate of growth of income — in the US since the mid-1980s. This negative or zero correlation between wealth inequality and r-g is also observed for other measures, for longer time periods, and in Sweden (the only other country we have appropriate wealth inequality measures for). Our wealth inequality data comes from the Survey of Consumer Finances (SCF), as summarized Table 2 of Wolff (2010). We show the Share of Total Wealth of the Top 1% (5 & 10) of Wealth. Data on the difference between the rate of return to wealth (r) and the growth rate of income is measured in two ways. One, following the literature standard, is as the difference between 3-month Treasury bills and the nominal GDP growth rate (the difference in nominal rates equaling the difference in real rates, under the assumption that the same nominal price deflator is appropriate for both). The other measures r as the ratio of the Capital Share of Output (rK/Y) divided by the Wealth-Income ratio (K/Y), using data from Karabarbounis and Neiman (2014) and WWID respectively, and then subtracts from this the real GDP growth rate.



* Data on wealth inequality from the Survey of Consumer Finances (SCF), as summarized Table 2 of Wolff (2010). We show the Share of Total Wealth of the Top 1% (5 & 10) of Wealth. Data on the difference between the rate of return to wealth (r) and the growth rate of income is measured in two ways. One is as the difference between 3-month Treasury bills and the nominal GDP growth rate. The other measures r as the ratio of the Capital Share of Output (rK/Y) divided by the Wealth-Income ratio (K/Y), using data from Karabarbounis and Neiman (2014) and WWID respectively, and then subtracts from this the real GDP growth rate.

Figure 1: Negative or Zero Correlation between Wealth Inequality and r-g in US since mid-1980s.

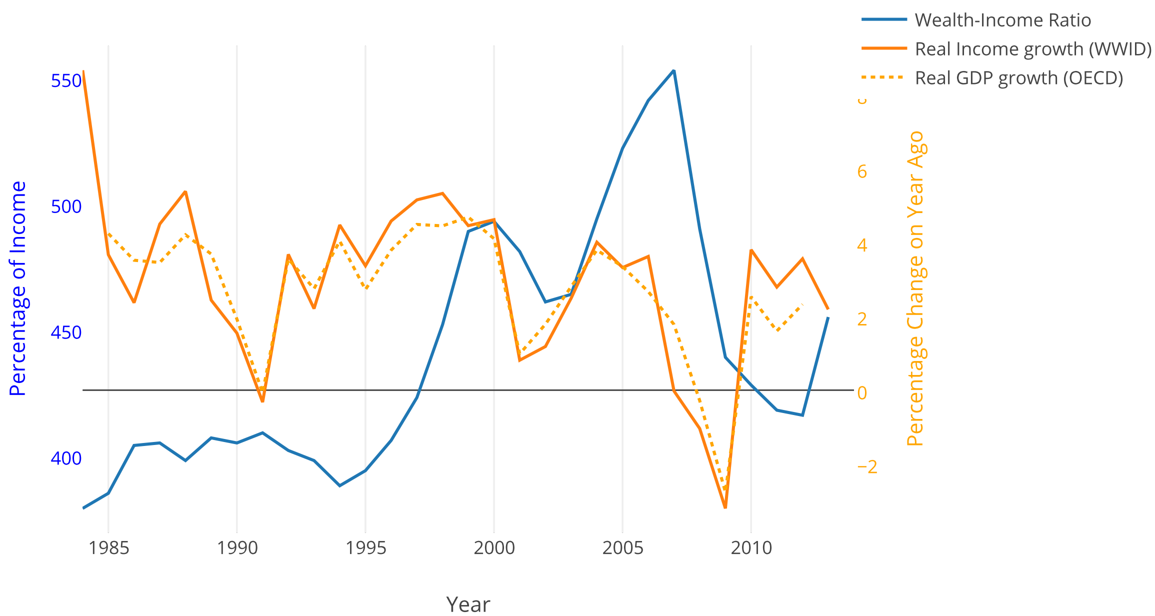
Appendix A shows that these results are robust to the use of the longer time periods available if we instead use the wealth inequality numbers of Saez and Zucman (2016). It also shows that they are robust to using different deflators for interest rates and GDP growth (ie. CPI inflation and the GDP deflator), different interest rates (eg. 10yr), and different data sources for the capital share of output (OECD).

The only other country we consider is Sweden, to which our results are robust (Appendix A), all other countries are ignored simply due to lack of data on wealth inequality. Some wealth inequality data does exist for France and the UK, but is based on estate tax data and so we choose not to use it as we do not think it is of the same quality, for reasons discussed in the introduction.

Since in the model changes in r-g would involve movements in both an endogenous variable and an exogenous variable we would be unable to provide analytical results on it's behaviour. While it remains very much of interest we therefore now turn to instead characterizing four correlations which we can address both in the data and in the model. We return to further discussion of r-g

and Wealth Inequality in Section 4

Correlation (i): Negative Correlation between Wealth-Income ratio and rate of income growth (g):



* Data on (Net National) Wealth-Income ratio and Income growth rate (calculated as percentage change in income) from World Wealth and Income Database (WWID). For comparison OECD data on growth rate of real GDP is also included.

Figure 2: Negative Correlation between Wealth-Income ratio and rate of income growth (g) in US since mid-1980s.

Figure 2 shows the negative correlation between Wealth-Income ratio and the rate of income growth observed in the US economy since the mid-1980s. This negative correlation is weak but present in the US both over this time period and over the longer time periods for which data is observable. The weakness of the correlation is complicated by the lack of any substantial and persistent changes in the rate of real income growth observed in the US. When longer time periods are used the most obvious movements in the Wealth-Income ratio for the US relate to the World Wars, which the model would clearly fail to capture. This negative correlation between the Wealth-Income ratio and the growth rate of real income we observe for the US since the mid-1980s is also present in most of the other countries we looked at (all countries with available data show large movements relating to the world wars). Our result would robust to using the capital-output ratio instead of the wealth-income ratio.

Correlation (ii): Negative Correlation between Wealth Inequality and rate of return

to wealth (r): Figure 3 shows the negative correlation between Wealth Inequality and the rate of return to wealth (r) present in the US economy since the mid-1980s. We present two measures of the rate of return to wealth (as previously discussed). The first is the real interest rate measured as the difference between the 3-month interest rate on Treasury bills and the CPI inflation rate. As discussed in the introduction, while standard in both the literature on r-g and on secular stagnation, it is not clear that this is the appropriate measure. We therefore also consider a more direct measure of the return to wealth: observing that in a neoclassical growth model[17] the capital share of output is rK/Y, and the wealth-income ratio is K/Y, so we can measure the rate of return to wealth as r equals capital share of output divided by the wealth-income ratio. Our other measure does this using the capital share of output data of Karabarbounis and Neiman (2014)[18] and the wealth-income ratio data of the WWID. Appendix A looks at other measures, and time periods, finding this negative correlation is robust to using other interest rates (eg. 10yr) and other methods to calculate the real interest rate (eg. GDP deflator, rather than CPI inflation[19]); and to using other wealth inequality measures, together with longer time periods where feasible (the longest we consider are 1954 to the present). Our results would not be robust to the use of the rate of return to capital of Gomme et al. (2011), but this is the sole exception, and suggests that the emphasis of Blume and Durlauf (2015) on the distinction between wealth and capital is important in practice as well as theory. As previously discussed in relation to r-g and wealth inequality, we do not here consider robustness to the use of other countries due to the paucity of wealth inequality data.

Correlation (iii): Positive Correlation between Income Inequality and Wealth Inequality:

Figure 4 shows the positive correlation between income inequality and wealth inequality observed in the US since the mid-1980s. We use household level data from the SCF. Appendix A shows our results are robust to the use of tax-unit level data and longer time periods (although as discussed in the Introduction we prefer the SCF data as dynastic households provide the 'model-consistent' interpretation of inequality data). We do not look at this correlation in other countries, partly because of a lack of suitable wealth inequality data in almost all countries, and partly as a positive correlation between income and wealth inequality seems rather uncontroversial.

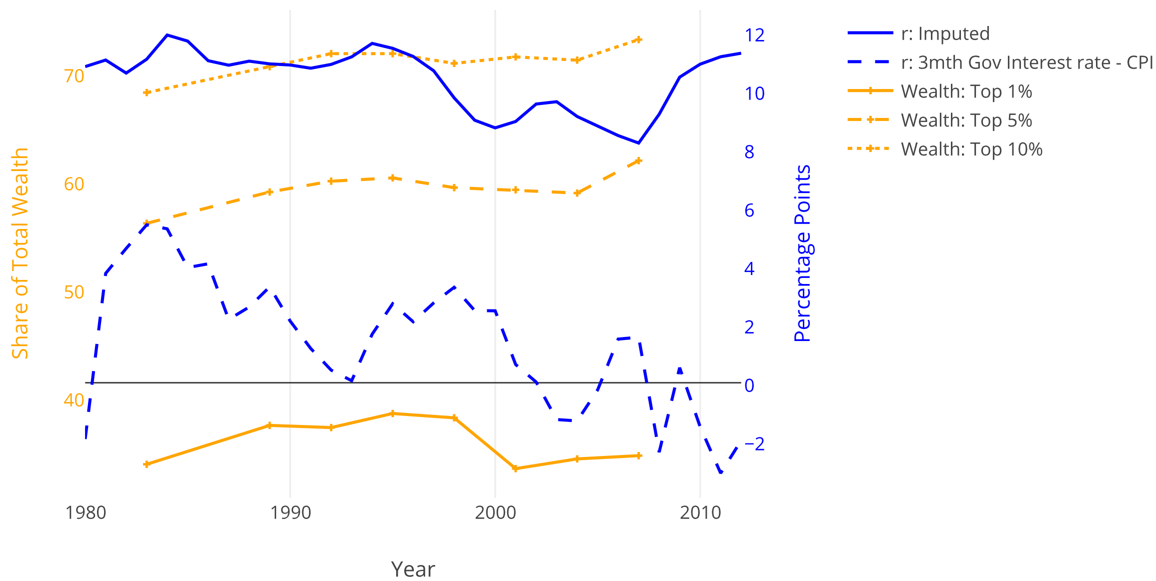
A tighter link between data and model would require that we instead look at the correlation between Earnings (income from labour) Inequality and Wealth Inequality. We do not do so purely for reasons of data availability; Appendix A shows, for the limited available data, that it would not change our conclusions.

Correlation (iv): Positive Correlation between Income Inequality and Income Risk:

[17] Including that with heterogeneous agents and incomplete markets which is presented in this paper.

[18] Technically, 1 minus their labour share of output data.

[19] Real interest rates based on TIPS ('inflation protected') treasury series are not checked as the time series is too short,



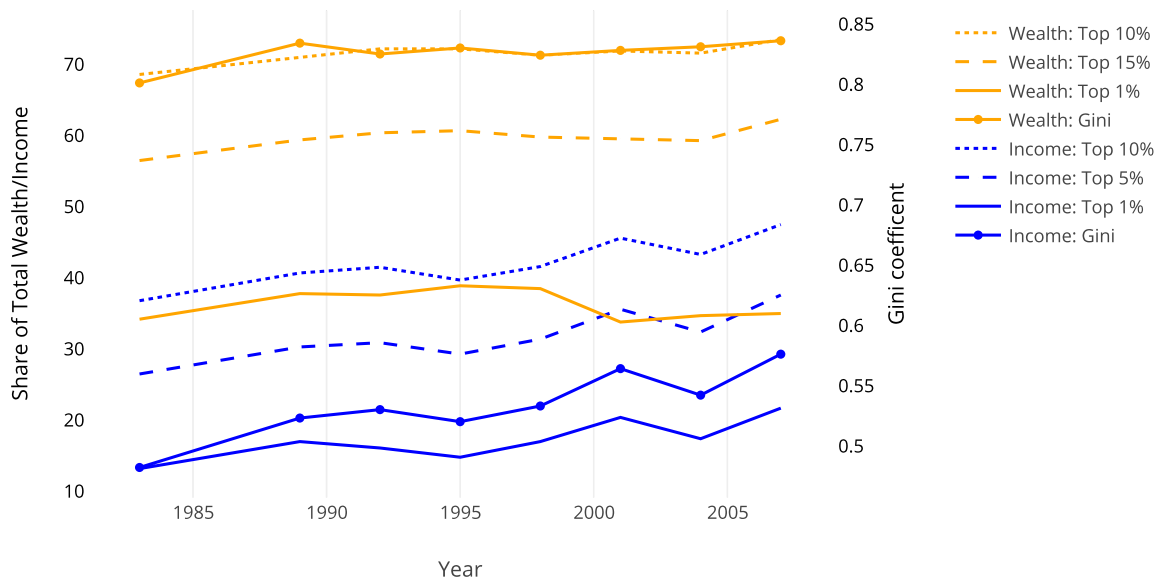
* Data on wealth inequality from the Survey of Consumer Finances (SCF), as summarized Table 2 of Wolff (2010). We show the Share of Total Wealth of the Top 1% (5 & 10) of Wealth. Data on the rate of return to wealth (r) is measured in two ways. One, following the literature standard is as the difference between 3-month Treasury bills and the CPI inflation rate. The other is as the ratio of the Capital Share of Output (rK/Y) divided by the Wealth-Income ratio (K/Y), using data from Karabarbounis and Neiman (2014) and WWID respectively.

Figure 3: Negative Correlation between Wealth Inequality and rate of return to wealth (r).

Figure 5 shows the positive correlation between income inequality and income risk. Our US data on income risk measured as the ratio of (or difference between) the variance of income to the variance of consumption comes from the Consumer Expenditure Survey, as summarized in Heathcote, Perri, and Violante (2010). Fisher, Johnson, and Smeeding (2015) suggest that we would get similar results for 1984-2011 if we use a wider demographic (rather than just the working-age white males used in the Heathcote, Perri, and Violante (2010) numbers).

Our interest is in differentiating between two possibilities: that the increase in inequality reflects a divergence between different 'fixed types' of people, or that the increase in income inequality reflects an increased variance of 'shocks' to individuals incomes.[20] This matters because in the

[20] A simple example better illustrates the meaning of the difference: Say there are two people, person A and person B. To begin with both people have incomes of one-unit every period. The first case of a divergence between different 'types' might be thought of as the case where person A now has income of two-units per period, while person B has one-unit; so if we look at the cross-section we observe income inequality of $(2, 1)$, and consumption inequality of $(2, 1)$. The second case of an increased variance of income shocks might be thought of as the case where person A and person B now both have alternating incomes of two-units and one-unit in a two-period cycle; so if we look at the cross-section we observe income inequality of $(2, 1)$, and consumption inequality of $(1.5, 1.5)$ (assuming perfect consumption smoothing).



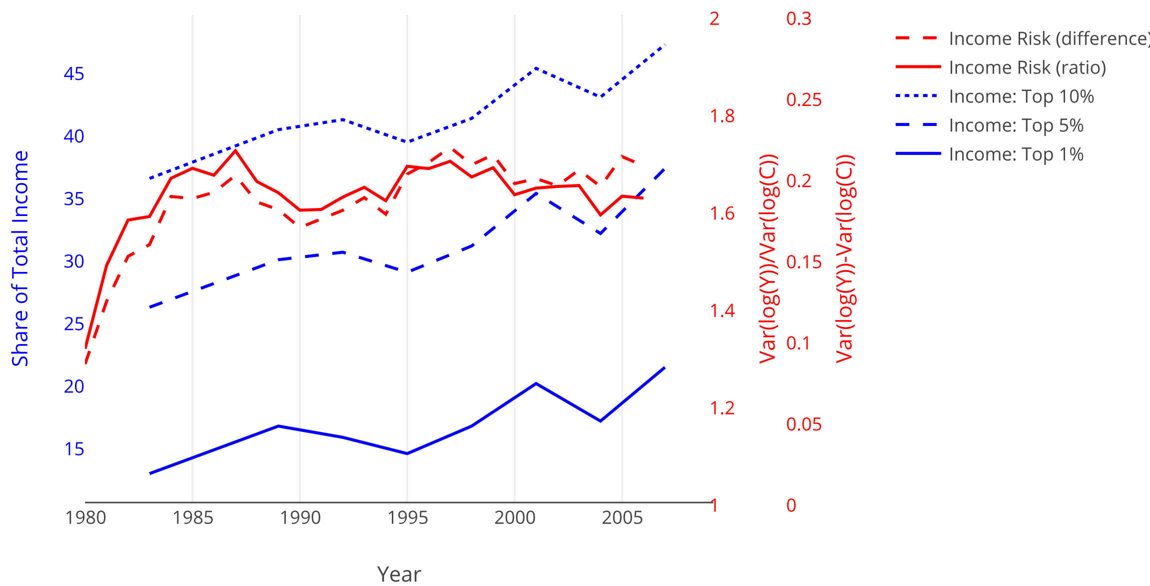
* Data all taken from the Survey of Consumer Finances (SCF), as summarized Table 2 of Wolff (2010). We show the Gini coefficients for income and wealth. We show the Total Income (Wealth) shares of the Top 1% of Incomes (Wealth); note that while there is overlap these are not the same households.

Figure 4: Positive Correlation between Income Inequality and Wealth Inequality in the US since the mid-1980s.

neoclassical growth model with heterogeneous agents and incomplete markets the increase in income inequality reflects the second type — increased variance of shocks to incomes, and this constitutes one of the important differences between this model and the kind of model of wealth inequality favoured by Piketty (2014) (see Appendix E).

In the first case of 'fixed types' we would expect to see consumption inequality rise by the same amount as income inequality, in the second case we would expect a smaller rise in consumption than in income inequality. This suggests that some measure of the relative changes in income inequality and consumption inequality would be a suitable measure of income risk. We present two: the ratio of the variance of log income to the variance in log consumption, and the variance of log income minus the variance of log consumption. Both show an increase in income risk, but differ in when it occoured.

This measure is not without issues. The main one is that the effects of income risk are deeply intertwined with the availability of insurance and seperating the two requires some assumptions; Guvenen and Smith (2014) provide one approach to this, but do not look at changes in income risk over time. While our measures do not account for any changes in the availability of (self-



* Data on income inequality from the Survey of Consumer Finances (SCF), as summarized Table 2 of Wolff (2010). We show the Total Income shares of the Top percentiles of Income. Data on Income Risk is from the Consumer Expenditure Survey (CEX), as summarized in Heathcote, Perri, and Violante (2010).

Figure 5: Positive Correlation between Income Inequality and Income Risk.

or market) insurance we are confident in concluding that while they do not provide an accurate picture of 'how much' income risk has increased they do capture the direction of the change. This confidence mostly follows from an extensive microeconometric literature that finds similar trends of increasing income risk using a variety of different methods Arellano, Blundell, and Bonhomme (2015); Attanasio and Pistaferri (2014); Krueger, Perri, Pistaferri, and Violante (2010); Moffitt and Gottschalk (2009, 2012).

We close by observing that in reality increasing income inequality is likely better captured as a combination of these two concepts: the 'fixed types' and the 'variance of shocks'. This can be trivially done in the neoclassical growth model with heterogeneous agents and incomplete markets simply by adding a finite number of types of agents and would not affect our analytical results (in the sense that an increase in the variance of shocks would still have the same effect, given fixed types; the interaction between changes in the variance of shocks and changes in fixed types could be analysed quantitatively, but not analytically).

3 Neoclassical Growth Model with Heterogeneous Agents

I now introduce and analyse the Neoclassical Growth Model with Heterogeneous Agents, Borrowing Constraints and Incomplete Markets. I provide new analytical results showing that it predicts a negative relationship between inequality and the interest rate r. The model is a neoclassical growth model with a continuum of households facing idiosyncratic but no aggregate shocks, who make decisions about consumption and savings. [21] Economic growth occours deterministically at constant rate g. Markets are incomplete in the household dimension (there are no assets whose return is conditional on the idiosyncratic shocks), and there are borrowing constraints (individual assets must be non-negative). Huggett (1997) introduced this model but his analysis focuses on dynamics and steady-states, and does not consider balanced growth paths nor the cross-sectional distribution.

In Section 3.1 I show that the relationships between inequality, income risk, and r-g on Balanced Growth paths in the neoclassical growth model with heterogeneous agents and incomplete markets can be analysed by looking at the stationary equilibria of a growth-adjusted model. Two main analytical results for the growth-adjusted model are shown, namely the negative relationship between income risk and interest rates (Section 3.2), and the negative relationship between economic growth g and interest rates r (meaning that a focus of r-g alone would be misleading; Section 3.3).

After deriving general these analytical results I perform some quantitative exercises to understand more fully the relationship between income risk, inequality, r, and g.

Analysing the model depends on a combinations of original results and drawing on various sources in the literature. The balanced growth path aspects build on Cooley and Prescott (1995) & Krueger and Lustig (2010). In the cross-sectional dimension the model looks similar to a generalization of Aiyagari (1994). Analysing inequality and interest rates extends various results from Clarida (1987, 1990), Huggett (2003, 2004), Jensen (2014) and Acemoglu and Jensen (2015); although none of these papers focus on the model used here, nor on the relationship between inequality and interest rates.

Original contributions in this section include the introduction of the definition of a Balanced-Growth-Path for these models, and results showing that the behaviour of prices, growth rates, and inequality along this Balanced-Growth-Path can be analysed in terms of the stationary equilibria of a growth-adjusted economy. The joint analysis of inequality, income risk, and r-g in these models is also new.

While the model used in the quantitative section generates lower levels of inequality than those seen in the data, more advanced versions of these models are capable of reproducing realistic levels of inequality (Castaneda, Díaz-Giménez, and Ríos-Rull, 2003) and the extension of such a model

[21] Appendix D.2 provides an extension of all the results to include endogenous labour supply.

to include real output growth would be covered by the analytical results.

3.1 Model Environment and Equilibrium Definitions

The model is a one-sector neoclassical growth model with a continuum of households facing idiosyncratic but no aggregate shocks, with incomplete markets (and borrowing constraints), who make decisions about consumption and savings. The model is a variation of Huggett (1997), and when focusing on the balanced growth path it simplifies to a variation of Aiyagari (1994). I begin by defining the model economy and introducing a definition for a Balanced Growth Path. A growth-adjusted economy is then defined, a Stationary Equilibrium of which is shown (Appendix C)to be equivalent to a (renormalized) Balanced Growth Path of the original economy. The results of this section imply that

Model Environment: In the model infinitely lived households face a combination of an idiosyncratic stochastic process with labour supply, z_t, and a deterministic growth of labour efficiency units, E_t. Denote a history of idiosyncratic shocks by $z^t = \{z_t, ..., z_1, z_0\}$. The households then make decisions on consumption and savings; given an interest rate path, $\{r_t\}$ and an wage path $\{w_t\}$.

The households order sequences of consumption $\{c_t(z^t, t)\}$ (and implicitly capital holdings $\{k_{t+1}(z^t, t)\}$) according to the utility function

$$U(\{c_t(z^t)\}) \equiv \sum_{t=0}^{\infty} \int_{z^t \geq z_0} \beta^t Q^t(z_0, dz^t) \frac{c_t(z^t)^{1-\gamma}}{1-\gamma} \tag{1}$$

The dependence of the utility function U on initial conditions, E_0 and an initial agent distribution, has been supressed in the notation as it will play no role. The idiosyncratic labour supply shock z_t is assumed to follow a stationary 1st-order Markov process with transition function $Q(z, dz')$. $Q^t(z^0, dz^t)$ is the t-step transition function defined iteratively by $Q^t(z_0, dz^t) = \int_{z^{t-1} \geq z_0} Q(z_{t-1}, dz_t) Q^{t-1}(z_0, dz^{t-1})$, from the stationary 1st-order Markov transtion function $Q(z, dz')$.

The households problem is to maximize utility subject to a budget constraint and a borrowing constraint,

$$\max_{\{c_t(z^t), k_{t+1}(z^t)\}} U(\{c_t(z^t)\}) \tag{2}$$

$$s.t. \quad c_t(z^t) + k_{t+1}(z^t) = w_t z_t E_t + (1 + r_t) k_t(z^{t-1}), \quad \forall z^t, \forall t \tag{3}$$

$$k_{t+1}(z^t) \geq \underline{\mathrm{k}}(E_t), \quad \forall z^t, \forall t \tag{4}$$

$$k_1 \text{ given} \tag{5}$$

The economic growth of the economy is driven by E_t which grows deterministically according to $E_{t+1} = (1 + g)E_t$. Observe that the budget constraint, $c_t + k_{t+1} = w_t z_t E_t + (1 + r_t) k_{t+1}$, other than the stochastic income and the absence of a wealth tax, is essentially the same as equation (41) which formed the basis of the *r-increases-inequality* model.[22]

Individual household capital holdings, given an interest rate, aggregate to give aggregate capital holdings. The market clearance condition is that the interest rate will be determined by perfect competition in the goods market together with a representative firm with Cobb-Douglas production function and a constant rate of labour-augmenting technological growth.

All the different types of equilbrium definition considered below are general equilibria definitions for this model, and all involve perfect competition. I start by introducing a definition for a Balanced Growth Path equilibrium, and then give the standard definition of a Stationary equilibrium.

In an economy such as this with heterogeneous agents a Balanced Growth Path is defined based on the concept that prices are constant and that the distribution of agents is invariant if considered on a renormalization of agents state-space by the aggregate state.

Definition 1. *A Balanced Growth Path is an agents utility function U and allocation decisions $\{c_t(z_t), k_{t+1}(z_t)\}_{t=0}^{\infty}$; interest rate and wage $\{r, w\}$; aggregate capital and labour $\{K_t, L_t\}_{t=0}^{\infty}$; and a measure of agents $\{\mu(k, z)\}_{t=0}^{\infty}$; such that*

1. *Given prices $\{r, w\}$, the agents utility function and allocation decisions solve the agents problem given by (1)-(5)*
2. *Aggregates are determined by individual actions: $K_t = \int k_t(z^t) d\mu_t(k_t(z^t), z^t)$, and $L_t = \int z^t E_t d\mu(k_t(z^t), z^t)$*
3. *Markets clear (in terms of prices): $r - (\alpha K_t^{\alpha-1} L_t^{1-\alpha} - \delta) = 0$ and $w - (1-\alpha) K_t^{\alpha} L_t^{-\alpha} = 0$ for all t.*
4. *The measure of agents is generated by the allocation decisions and is invariant on a renormalization of the state-space by the aggregate endowment:*
$$\mu(k_t(z^t)/E_t, z^t) = \int \int_{z^t \geq z^{t-1}} (k_t(z^{t-1})/E_t, z^{t-1}) \mu(k^{t-1}/E_{t-1}, z^{t-1}) Q(z_{t-1}, dz_t) \tag{6}$$

where z is the labour supply shock which evolves according to Markov transition function $Q(z, dz')$. Notice that μ is time-invariant when considered on the space $(k^t/E_t, z^t)$, rather than (k^t, z^t) (this is important for the concept of a balanced-growth path).

The standard definition of a Stationary (Competitive) equilibrium is given by,

Definition 2. *A Stationary Equilbrium is an agents utility function U and allocation decisions $\{c_t(z^t), k_{t+1}(z^t)\}_{t=0}^{\infty}$; sequences for interest rate and wage r, w; aggregate capital and labour K, L; and a measure of agents $\mu(k, z)$; such that*

[22]The budget constraint is also time dependent, but the time dependence will disappear when we focus on the Balanced Growth Paths.

1. *Given prices r, w, the agents utility function and allocation decisions solve the agents problem given by (1)-(5)*

2. *Aggregates are determined by individual actions: $K = \int k_t(z^t) d\mu(k_t(z^t), z^t)$, and $L = \int z^t E_t d\mu(k_t(z^t), z^t)$*

3. *Markets clear (in terms of prices): $r - (\alpha K^{\alpha-1} L^{1-\alpha} - \delta) = 0$.*

4. *The measure of agents on the state-space is generated by the individuals allocation decisions:*

$$\mu(k_t(z^t), z^t) = \int \int_{z^t \geq z^{t-1}} (k_t(z^{t-1}), z^{t-1}) \mu(k^{t-1}, z^{t-1}) Q(z_{t-1}, dz_t) \tag{7}$$

where z is the labour supply shock which evolves according to Markov transition function $Q(z, dz')$. Notice that μ is now time-invariant and no renormalization is performed. The model economy with exogenous labour supply clearly does not have a Stationary equilibrium itself, but the concept will be useful in the growth-adjusted version of the model which is now introduced.

As shown in Appendix C there is mapping between Balanced Growth Paths of the original model and stationary competitive equilibria of a growth-adjusted economy. This mapping is based on the definition of the growth-adjusted economy in terms of the hatted variables given by

$$\hat{c}_t \equiv c_t / E_t \tag{8}$$

$$\hat{\beta} \equiv \beta(1+g)^{1-\gamma} \tag{9}$$

$$\hat{k}_t(z^{t-1}) \equiv k_t(z^{t-1}) / E_t \tag{10}$$

$$\hat{\underline{\mathrm{k}}}_t(z) \equiv \hat{\underline{\mathrm{k}}}_t(z) / E_{t+1} \tag{11}$$

$$\hat{w}_t \equiv w_t,\ \hat{r}_t \equiv r_t \tag{12}$$

$$\hat{K}_t \equiv K_t / E_t,\ \hat{L}_t \equiv L_t / E_t \tag{13}$$

$$\hat{\mu}_t(\hat{k}, z) \equiv \mu_t(k / E_t, z) \tag{14}$$

for $t = 0, 1, 2, ...$, scaled to recover an allocation in the original growth model by $E_t = (1+g)^t E_0$, $t = 0, 1, 2, ...$, where E_0 is initial aggregate labour productivity units. In the growth-adjusted economy the households decision problem, originally equations (1)-(5), is given by

$$\hat{U}(\{\hat{c}_t(z^t)\}) \equiv \sum_{t=0}^{\infty} \int_{z^t \geq z_0} \hat{\beta}^t Q^t(z_0, dz^t) \frac{\hat{c}_t(z^t)^{1-\gamma}}{1-\mu}$$

together with

$$\max_{\{\hat{c}_t(z^t), \hat{k}_t(z^t)\}} \quad \hat{U}(\{\hat{c}_t(z^t)\}) \tag{15}$$

$$s.t. \quad \hat{c}_t(z^t) + (1+g)\hat{k}_{t+1}(z^t) = w_t z^t + (1+r_t)\hat{k}_t(z^{t-1}), \quad \forall z^t, \forall t \tag{16}$$

$$\hat{k}_{t+1}(z^t) \geq \hat{\underline{\mathrm{k}}}, \quad \forall z^t, \forall t \tag{17}$$

$$\hat{k}_1 \text{ given} \tag{18}$$

In what follows I analyse the relationship between between interest rates, growth rates and inequality on a Balanced Growth Path by looking at properties of stationary equilibria of the growth-adjusted economy. Importantly notice that the transformations of c, k, asset income, and wage earnings are such that standard inequality measures like Lorenz curves, Gini coefficients, and the shares of quintiles or percentiles will be exactly equal on the Balanced Growth Path to their values in the growth-adjusted economy; likewise for the measures of income risk.[23]

3.2 Effects of Increased Income Risk

In this section I show analytically that these model predict a negative relationship between inequality and r-g. (that a mean-preserving spread of the idiosyncratic shock process will decrease the equilibrium interest rate r). These analytical results are derived under the assumption of a fixed g. Since the growth-adjustment introduces the growth rate g into the households problem in multiple places (the discount factor and budget constraint) it is not analytically clear how these interact and so the role of g is investigated by computation. Note that this already introduces a difference with the earlier *r-increases-inequality* models in which what matters for wealth inequality was r-g while here g will play a role beyond it's direct effect on r-g (to put it another way, here r and g will play seperate roles).

The models potentially have multiple equilibria. In this section I show that for the majority of equilibria we will have that r-g is decreasing in inequality (that a mean-preserving spread of the idiosyncratic shock process will decrease the equilibrium interest rate r). This ignores the possibility that in theory an equilibrium may be created or destroyed by the mean-preserving spread of the idiosyncratic shock process. Kirkby (2016) provides a computational algorithm for solving Bewley-Huggett-Aiyagari models, such as this one, and proves that the algorithm will converge asymptotically (in the grid size and tolerance value for convergence) to all equilibria of the model; application of this algorithm finds that for a variety of parameter values there is always only one single equilibria, with no creation or destruction of equilibria. While this does not analytically prove that the model used here has a unique equilibrium it does make it highly probable and so the approach taken here seems most relevant even though it ignores the possibility that equilibria are created or destroyed by the mean-preserving spread.

I proceed by first establishing sufficient conditions on the models which will give an inverse relationship between r and inequality; with the causation running from increased inequality to decreased r. In these models inequality directly reflects income risk. To do I focus first on the behaviour of the models ignoring the general equilibrium condition. The models can be thought of as a mapping from r to $\tilde{r}$, and the general equilibrium condition is then that $r = \tilde{r}$.

[23]This follows directly from a combination of the definition of the growth-adjusted economy variables with the deliberate invariance of standard inequality measures to average incomes/wealth/etc.

The result is built by the following combination of steps: for given r, a mean-preserving spread of the exogenous shock process (an increase in income risk) leads to an increase in the distribution of agents (ordered by the increasing-convex stochastic order, needed to order mean-preserving spreads), which implies an increase in the aggregate capital stock, which implies a decrease in the interest rate $\tilde{r}$. This result is established for an arbitrary r, and so we thus have that the mapping from $r \mapsto \tilde{r}$ is decreasing in the mean-preserving spread.

Huggett (2003) shows that the key to the aggregate capital stock increasing in response to a mean-preserving spread of the income risk shock is convexity of the savings function. Jensen (2014) provides conditions on the return function of the value function problem under which the savings function will be convex.

The general result underlying the analysis of how a mean-preserving spread in the exogenous shock process results in an increase in the distribution of agents is given by,[24]

Theorem 1. *(Huggett, 2003; Theorem 1) Assume that* $\succeq$ *is a reflexive and transitory binary relation on* $\Lambda(S,(S)$, $\geq_\Theta$ *is a reflexive binary relation on* Θ*, and* $T_\Theta : \Lambda(S,(S) \to \Lambda(S,(S)$ *for all* $\theta \in \Theta$*. Then,*

(i) For all $\theta, \theta' \in \Theta$*, for all* $\lambda, \lambda' \in \Lambda(S,\mathcal{S})$*, it holds that* $\theta \geq_\Theta \theta'$ *and* $\lambda \succeq \lambda' \implies T_\theta \lambda \succeq T_{\theta'} \lambda'$.
if and only if

(ii) assumptions (A1) and (A2) hold, where

(A1): T_θ *increases in* θ*:* $\forall \theta, \theta' \in \Theta$, $\theta \geq_\Theta \theta' \implies T_\theta \lambda \succeq T_{\theta'} \lambda. \forall \lambda \in \Lambda(S,\mathcal{S})$

(A2): T_θ *is order preserving:* $\forall \lambda, \lambda' \in \Lambda(S,\mathcal{S})$, $\lambda \succeq \lambda' \implies T_\theta \lambda \succeq T_\theta \lambda'. \forall \theta \in \Theta$

Simplifying to the case in which the policy variable, endogenous state, and exogenous state we get the following: The expected value of a policy variable increases in response to a mean-preserving spread of the exogenous shock if the optimal policy be increasing and convex in the endogenous state, and increasing in the exogenous shock.

Theorem 2. *(Simplified version of Acemoglu & Jensen (2015) Theorem 8; Extension of Huggett (2004) Theorem 1 to Markov processes) Let* $g(x,z)$ *be the optimal policy. Assume that* T *is defined as the adjoint operator of* $g \cdot Q$*, and that stochastic dominance is the binary relation on distributions. If the following regularity conditions hold,*

1. $g(x,z)$ *is increasing and convex in* x.

2. $g(x,z)$ *is increasing in* z.

Then the stationary distribution of the policy variable ($\int_{X \times Z} g(x,z)\lambda(dx,dz)$*, where* λ *is the stationary distribution* $\lambda = T\lambda$*) is increasing in response to a mean-preserving spread of the exogenous shock.*

[24] Theorem 1 of Huggett (2003) is more general than the version, adapted to the infinite time horizon, given here.

So we have that if the savings function is convex, then the aggregate capital stock (which is equal to the mean value of savings) will increase (and so r decreases) in response to a mean-preserving spread of the income risk shock. We now turn to showing that the savings function is convex. For this we will need the following Theorem from Jensen (2014), but first we lay out two assumptions that it will need.

Assumption 1. *$\Gamma : X \times Z \to Y \subset X$ is non-empty, compact-valued, continuous, and has a convex-graph, ie. for all $x, \tilde{x} \in X$, $z \in Z$, $\lambda \in [0,1]$: $\lambda y + (1-\lambda)\tilde{y} \in \Gamma(\lambda x + (1-\lambda)\tilde{x}, z)$ whenever $y \in \Gamma(x,z)$ and $\tilde{y} \in \Gamma(\tilde{x}, z)$.*

Assumption 2. *$F : X \times X \times Z \to \mathbb{R}$ is bounded and continuous, and $\beta \in (0,1)$. Furthermore, $F(x,y,z)$ is concave in (x,y) and strictly concave in y.*

The following theorem lays out the univariate case, but it can be extended to multivariate case (see Jensen (2014)

Theorem 3. *(Jensen, 2015; Theorem 11) Consider the stochastic dynamic programming problem*

$$V(x,z) = \sup_{y \in \Gamma(x,z)} \{F(x,y,z) + \beta \int V(y,z')Q(z,dz')$$

under the assumptions 1 & 2. Let $g : X \times Z \to X$ denote the policy function defined by

$$g(x,z) = \arg\sup_{y \in \Gamma(x,z)} \{F(x,y,z) + \beta \int V(y,z')Q(z,dz')$$

Assume that $F(x,y,z)$ is differentiable and satisfies the following upper boundary condition: $\lim_{y^n \nearrow sup\Gamma(x,z)} D_y F(x,$ $-\infty$ (or in some way ensure that $sup\Gamma(x,y)$ will never be optimal given (x,z)). Then the policy function g will be convex in x if $D_x F(x,y,z)$ is non-decreasing in y and there exists a $k \geq 0$ such that $\frac{1}{1-k}[-D_y F(x,y,z)]^{1-k}$ is concave in (x,y) and $\frac{1}{1-k}[D_x F(x,y,z)]^{1-k}$ is convex in (x,y). If in addition $\Gamma(x,\cdot)$ is a convex correspondence and $\frac{1}{1-k}[-D_y F(x,y,z)]^{1-k}$ is concave in (y,z), the policy function g will also be convex in z.

Remark: (Multi-dimensional choice sets): Theorem 3 can be extended to the case where X & Z are multidimensional. This requires the additional assumptions that F is supermodular in y, that Γ's values are lower semi-lattices, and that optimizers stay away from the upper boundary.

The conditions of Theorem 3 are satisfied by the HARA-class of utility functions. This includes the CES-utility functions common in Macroeconomics and which are now used in an examination of the quantitative magnitudes of the negative relationship between $r - g$ and inequality that is predicted by these models.

3.3 Effects of Increased Economic Growth

The role of g in the growth-adjusted model is both to decrease the patience of the growth-adjusted households (as $\hat{\beta} = \beta(1+g)^{1-\gamma}$) and to decrease the effectiveness of investment (the $(1+g)\hat{k}_{t+1}$

term in the budget constraint).

That a decrease in patience decreases aggregate capital and therefore increases r was shown (for iid case) in Clarida (1987) as part of the proof of his Theorem 3.1 (see his equation (44)).

So increased g implies a decrease in aggregate captial and an increase in r. Thus we cannot just look at $r - g$ when considering inequality, but have to consider both r and g seperately.

3.4 Full Interaction of Inequality and r-g

We have seen analytically that the *inequality-decreases-r* models predict a negative relationship between *r-g* and inequality; we now turn to putting some numbers to how strong this negative relationship might be.

Table 1 provides simulation results for various parameter values that give an idea of the predicted size of the relationship for the model without endogenous labour.

Table 1: 'r-g' and Inequality in Aiyagari (1994)

	$\sigma = 0.2$				
		Gini Coeffs		Inverted Pareto Coeff.	
ρ	Net Return to Capital in %	Income	Wealth	Income	Wealth
0.0	4.0669	0.11	0.37	0.53	0.42
0.3	4.0419	0.11	0.36	0.53	0.47
0.6	3.9421	0.11	0.38	0.58	0.56
0.9	3.5679	0.12	0.47	0.58	0.48
	$\sigma = 0.4$				
		Gini Coeffs		Inverted Pareto Coeff.	
ρ	Net Return to Capital in %	Income	Wealth	Income	Wealth
0.0	3.8673	0.22	0.34	0.51	0.50
0.3	3.5679	0.22	0.35	0.52	0.49
0.6	3.0689	0.22	0.38	0.54	0.49
0.9	2.0958	0.23	0.48	0.60	0.52

Aiyagari (1994) with μ=3 using grid sizes $n_k = 512$, $n_z = 21$, $n_p = 251$

3.5 The Model and the Four Correlations

4 'r-g' and Inequality

In *Capital in the Twenty-First Century* Thomas Piketty describes three laws-of-capitalism. The first is simply an accounting identity, that the capital share of income equals the return on capital times the capital-income ratio. The second, a relationship between savings, the capital-income ratio, and the rate of income growth, is almost[25] canonical macroeconomics, being a standard prediction of the Solow-Swan neoclassical growth model. The third law is that r-g — the interest rate minus the rate of income growth — is positively related to wealth inequality. It is this third law, the relationship between r-g and wealth inequality that we now address.

Acemoglu and Robinson (2015) provides a first-pass at some cross-country OLS on the relationship between r-g and income (not wealth!) inequality. They find no evidence that higher r-g leads to increased income inequality. They regress the Share of Total Income going to the Top 1% on r-g (the nominal interest rate minus the nominal growth rate). They look at annual data, allowing for up to four annual lags or r-g, and considering averages over the last 10 years and last 20 years of r-g. Their results (Table 1, Acemoglu and Robinson (2015)) show that the coefficients are mostly statistically insignificant, and on the occasions they are weakly statistically significant tend to be negative. The analysis is however limited by the focus on income inequality, rather than wealth inequality.

As well as emphasising that it is *wealth* inequality that is related to r-g, Piketty (2015) further emphasises that r-g is likely one of many factors driving inequality and so even if r-g is positively correlated with wealth inequality the relationship is likely to be conditional, rather than the unconditional relationship that Acemoglu and Robinson (2015) find is absent with income inequality.[26]

Hence empirical findings suggest that the relationship between r-g and wealth inequality is a negative or zero correlation.

As demonstrated by the neoclassical growth model with heterogeneous agents and incomplete markets it is perfectly reasonable that the relationship between r-g and wealth inequality is a negative correlation with the key being the role of income risk in inequality. We tentatively note that in terms of timing much of the rise in income risk that we saw in the US experience appears to occour roughly contempouraneously with much of the fall in real interest rates; namely in the 1980s.

We consider Piketty's third law-of-capitalism — that r-g, the rate of return to wealth minus the rate of income growth, is positively related to wealth inequality — to be falsified. Further we conclude that the correlation between r-g and wealth inequality is in fact often negative.

All of this is not to say that rates of return to wealth are irrelevant in driving wealth inequality.

[25] *Almost* as it is missing population growth, and more importantly depreciation.

[26] To deal with Piketty (2015)'s emphasis that r-g is just one of many factors we looked more closely at the traditionally Anglo-Saxon countries of Australia, Canada, New Zealand, UK, and USA, as an informal way to control for other factors; the idea being that institutional and cultural factors are comparatively similar across these countries. Due to data limitations this analysis only looked at the correlation between income (not wealth) inequality and r-g and found a negative or zero correlation in all countries but Canada. Interestingly Canada was also the only one in which our measures of income risk did not find an increase in income risk (although we had no numbers for New Zealand). Given it is just one country however this should be considered more as suggestive and worthy of further investigation rather than conclusive of anything. These results are not reported since we decided the use of income inequality as a proxy for wealth inequality was not a good way to proceed.

But it seems more likely to be related to their cross-section. Piketty's book suggests that wealthy have access to higher rate of returns, and presents data on university endowments that shows the mechanism is empirically valid in that specific case; Saez and Zucman (2016) show that the same is true of trust funds — larger funds have higher rates of return, and that it appears to be entirely due to portfolio effects. Benhabib, Bisin, and Zhu (2011, 2015) provide a theoretical models showing that stochastic interest rates could potentially be an important generator of inequality.

5 Conclusion

The outline of US experience with the Economy and Inequality since the mid-1980s can be broadly captured with four correlations: (i) a negative correlation between the rate of income growth and the wealth-income ratio; (ii) a negative correlation between wealth inequality and the rate of return to wealth; (iii) a positive correlation between income inequality and wealth inequality; and (iv) a positive correlation between income risk and income inequality.

These correlations, and the underlying trends they capture, speak to some important debates currently occouring in Economics: Secular Stagnation (Summers, 2014; Sec, 2014), implications of possible lower future income growth (Gordon, 2016), and rising inequality in income and wealth (Piketty, 2014).

That all four correlations arise jointly in the neoclassical growth model with heterogeneous agents and incomplete markets suggest first and foremost that they should not be treated in isolation from each other. In particular we emphasise our conclusion that Macroeconomics and Inequality are fundamentally linked.

It also suggests that the neoclassical growth model with heterogeneous agents and incomplete markets, while certainly not enough on its own to fully explain these issues, remains a useful benchmark for Macroeconomics and one capable of providing insight into important issues.

Since part of this paper has been spent criticizing Piketty's work we would like to conclude by repeating a demand of Piketty's with which we wholeheartedly agree, namely "A return of distribution to the center of the study of Economics!"... a demand that conveniently dovetails with our own Lucasian predeliction for microfoundations.

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A Other time periods, data sources, measurement concepts, and Countries

When we consider other countries we often look at Australia, Canada, and the UK, chosen both as they often have good data coverage and because as other historically Anglo-Saxon countries they act as an informal manner of controlling for institutions and culture. France, Germany, New Zealand, Spain, and Sweden are the other countries we tend to look at, chosen partly for data availability reasons, and partly just because, in the case of New Zealand and Spain, we find them of particular interest (New Zealand is often presented together with the traditionally Anglo-Saxon countries).

Negative or Zero Correlation between Wealth Inequality and r-g:

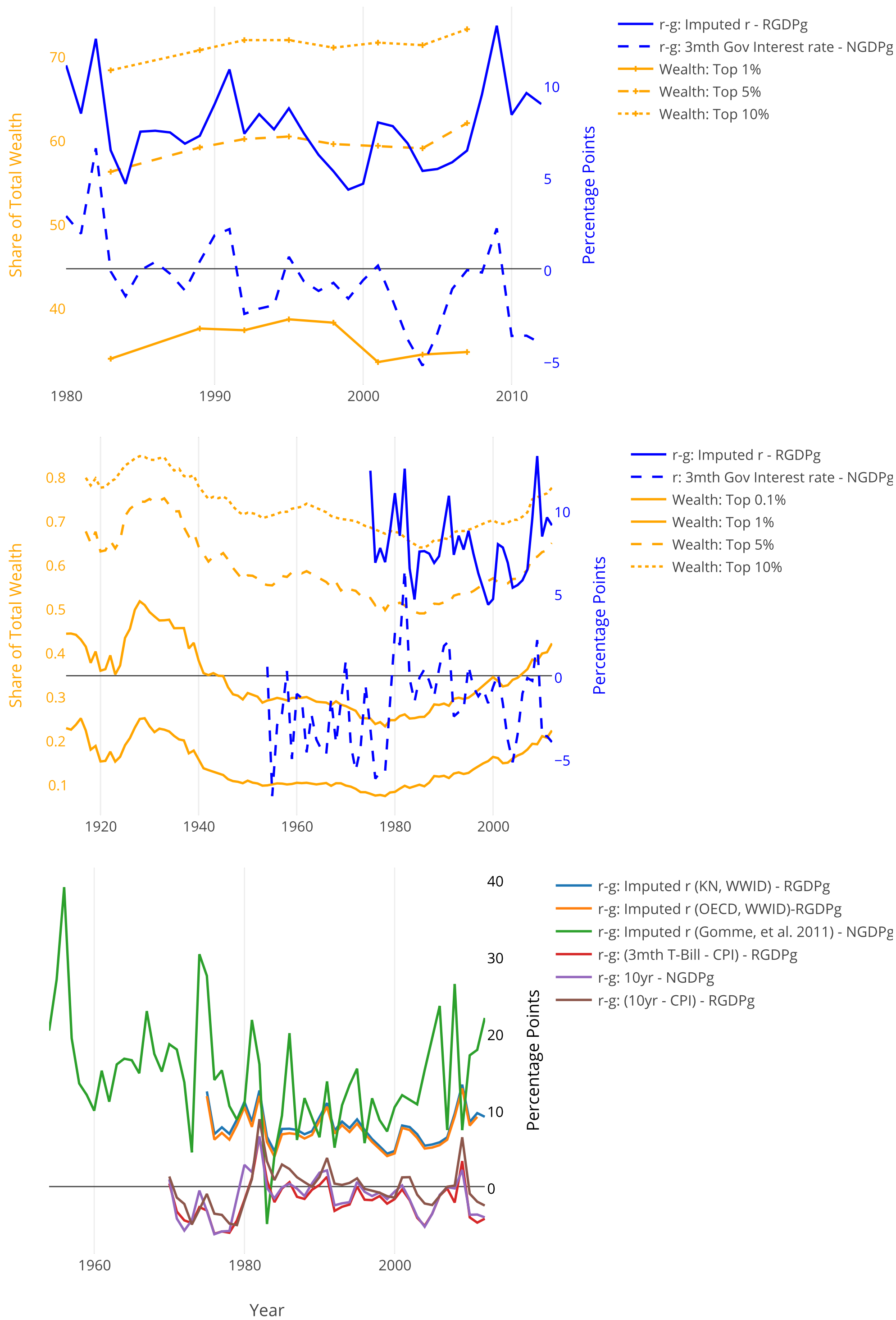
Table 2: Negative or Zero Correlation between Wealth Inequality and r-g in US
Correlation Coefficients between Wealth Inequality and r (for US)

	Wealth Inequality Measure		
	Share of Top 10% (SCF)	Gini Coeff. (SCF)	Share of Top 10% (SZ2016)
	[OLS coefficient (p-value)]		
Rate of Return to Wealth (KN,WWID)	0.2182 (0.6539)	0.0012 (0.7478)	0.2693 (0.3158)
Rate of Return to Wealth (OECD,WWID)	0.3238 (0.5113)	0.0019 (0.6167)	0.2625 (0.2988)
Rate of Return to Capital (GommeEtAl)	0.1376 (0.0906)	0.0014 (0.0169)	0.1546 (0.0060)
Interest rate (IMF 3mth, OECD CPI)	-0.0426 (0.8878)	-0.0004 (0.8572)	-0.3268 (0.0833)
Interest rate (IMF 3mth, OECD GDPDEF)	-0.1820 (0.5448)	-0.0017 (0.4666)	-0.3189 (0.1111)
Interest rate (FedRes 10yr, BLS CPI)	-0.2971 (0.4508)	-0.0034 (0.2468)	-0.4099 (0.0054)
Interest rate (OECD 24hrIB, OECD CPI)	-0.4581 (0.1834)	-0.0048 (0.0522)	-0.3471 (0.0261)

All regressions are run on the available annual data; note that SCF data is only periodic so contains few observations. The number of observations varies across and reflects all available data for each time series post-1954. Correlation coefficients are calculated by OLS regression (with a constant term) and p-values are for the standard null hypothesis of H0: coefficient=0. For purpose of regression the shares are measured as 0 to 100. In all cases NGDP or RGDP growth is subtracted as appropriate (ie. NGDP for rows 4 and 6). First data observations are 1975 for KN data, 1970 for OECD data, 1954 otherwise.

The top panel of Figure 6 shows the negative or zero correlation between wealth inequality and r-g — the difference between the rate of return to wealth and the rate of growth of income — in the US since the mid-1980s. This negative or zero correlation between wealth inequality and r-g is also observed for other measures, for longer time periods, and in Sweden (the only other country we have appropriate wealth inequality measures for). Our wealth inequality data comes from the Survey of Consumer Finances (SCF), as summarized Table 2 of Wolff (2010). We show the Share of Total Wealth of the Top 1% (5 & 10) of Wealth. Data on the difference between the rate of return to wealth (r) and the growth rate of income is measured in two ways. One, following the literature standard, is as the difference between 3-month Treasury bills and the nominal GDP growth rate (the difference in nominal rates equaling the difference in real rates, under the assumption that the same nominal price deflator is appropriate for both). The other measures r as the ratio of the Capital Share of Output (rK/Y) divided by the Wealth-Income ratio (K/Y), using data from Karabarbounis and Neiman (2014) and WWID respectively, and then subtracts from this the real GDP growth rate.

The second panel shows that these results are robust to the use of the longer time periods available if we instead use the wealth inequality numbers of Saez and Zucman (2016). The third panel shows r-g for many different measures of r using different deflators for interest rates and GDP growth (ie. CPI inflation and the GDP deflator), different interest rates (eg. 10yr), and different data sources for the capital share of output (OECD). Given that the same trends in r-g can be seen for these various measures our results are robust to their use.

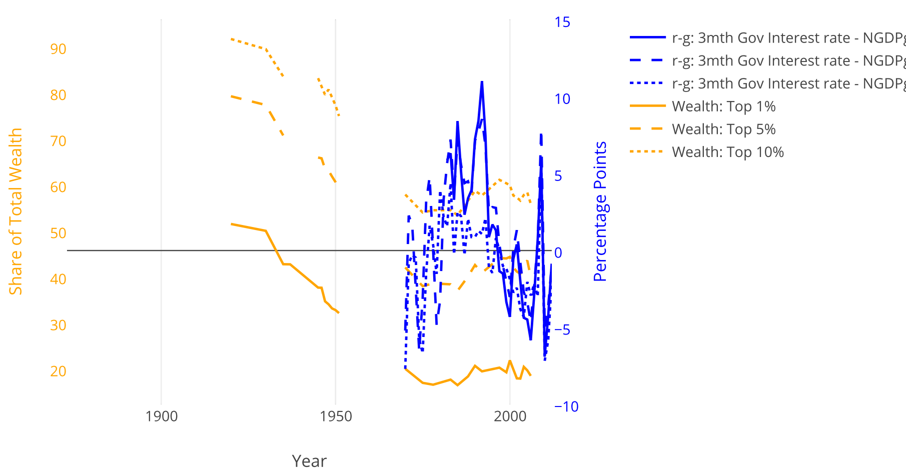


* The first panel shows data on wealth inequality from the Survey of Consumer Finances (SCF), as summarized Table 2 of Wolff (2010). We show the Share of Total Wealth of the Top 1% (5 & 10) of Wealth. Data on the difference between the rate of return to wealth (r) and the growth rate of income is measured in two ways. One is as the difference between 3-month Treasury bills and the nominal GDP growth rate. The other measures r as the ratio of the Capital Share of Output (rK/Y) divided by the Wealth-Income ratio (K/Y), using data from Karabarbounis and Neiman (2014) and WWID respectively, and then subtracts from this the real GDP growth rate. The second panel shows the same measures of r-g, for their full available time periods, together with tax-unit level measures of wealth inequality taken from Saez and Zucman (2016). The third panel shows r-g for a variety of alternative measures of r: the same two concepts r are used, but using different data (eg. 10yr instead of 3mth bonds, or OECD data on labour share of income); as well as using different deflators (CPI inflation vs GDP deflator) for r and g; all series not already used in earlier panels come from IMF and OECD (some of the data series in third panel do begin before 1970).

Figure 6: Negative or Zero Correlation between Wealth Inequality and r-g in US since mid-1980s.

Table 2 shows these negative or zero correlations presenting the correlation coefficients and their statistical significance for the various combinations of wealth inequality and r-g measures; all statistically significant results (those which measure r in terms of interest rates and using the longer time series of Saez and Zucman (2016)) are negative. Those which are not negative (those for our alternative measures of the rate of return to wealth) are statistically indistinguishable from zero (p-values are nowhere near statistical significance). The only exception is those relating to the return to capital of Gomme et al. (2011), suggesting the importance of the distinction between capital and wealth, as emphasised by Blume and Durlauf (2015) and which we discuss further later on. We conclude that the correlation coefficients agree with our graphical interpretation that the correlation between wealth inequality and r-g is negative or zero.

The only other country we consider is Sweden. The relation between Wealth Inequality and r-g for Sweden is shown in Figure 7. The negative correlation seems visually quite clear, and simple regressions find negative but statistically insignificant correlations (results not shown; eg. regressions of each of the three different r-g measures on the Top 10% wealth shares all yield negative but statistically insignificant coefficients). Hence it appears that the US experience is not unusual. All other countries are ignored simply due to lack of data on wealth inequality. Some wealth inequality data does exist for France (Piketty et al., 2006), Switzerland (Dell et al., 2007). Alvaredo et al. (2015) discuss the UK case and the difficulties faced in measuring the wealth distribution there, providing an example of why most countries lack such data. but is based on estate tax data and so we choose not to use it as we do not think it is of the same quality, for reasons discussed in the introduction.



* Wealth inequality data come Swedish Wealth Tax Data compiled in Table A1 of Roine and Waldenstrom (2009). The interest rates are nominal (24hr interbank, 3mth Treasury, and 2-to-9yr Treasury), with CPI inflation subtracted and then subtracting RGPD growth rates (all from OECD data).

Figure 7: Negative Correlation between Wealth Inequality and 'r-g' in Sweden

One other possibility would be take advantage of the positive correlation between income inequality and wealth inequality, and taking this empirical regularity as a direct fact we could indirectly assess by looking at the correlation between income inequality and r-g. Acemoglu and Robinson (2015) do this when looking at r-g and inequality: they focus on income inequality rather than wealth inequality. We performed this excerise of using income inequality as a proxy for wealth inequality for other countries, using both our main measures of r-g finding our results to be robust in other countries (correlations where almost all negative, though many were statistically insignificant). We do not report these results as we consider them to largely just repeating the findings of Acemoglu and Robinson (2015), and as using income inequality as a proxy for wealth inequality is clearly inferior where wealth inequality data is available.

Correlation (i): Negative Correlation between Wealth-Income ratio and rate of income growth (g):

We begin with the three panels of Figure 8 which show the negative correlation between Wealth-Income ratio and the rate of income growth observed in the US economy across the complete time periods for which the WWID data is available. The top panel shows the raw data and the other two panels present smoothed versions that allow us to more easily see the trends. The data series from the WWID are 'Net national wealth as percentage of national income', and national income is measures in the national currency with an index year that varies from 2010 to 2014 depending on the country (2013 for US). OECD data are the annual averages of the quarterly percent change on year ago of Constant Price Gross Domestic Product in 'Country'; which were accessed via FRED.

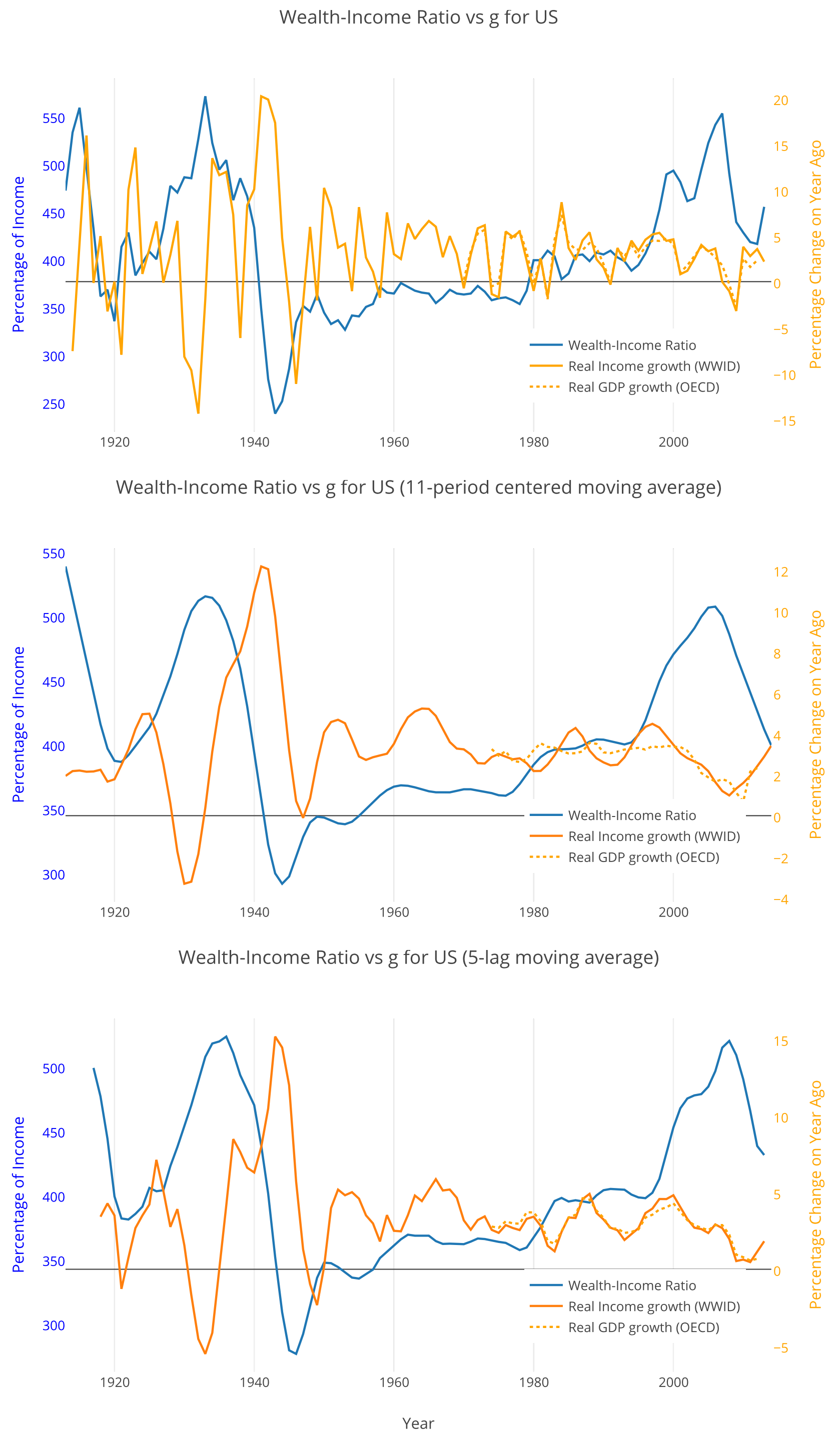
For this longer time period, the maximum for which data was available, we continue to observe the weak negative correlation we saw in the US since the mid-1980s. The weakness of the correlation is complicated by the lack of any substantial and persistent changes in the rate of real income growth observed in the US, other than the two World Wars, the Great Depression and the Great Depression. When longer time periods are used the most obvious movements in the Wealth-Income ratio for the US relate to the World Wars, which the model would clearly fail to capture.

Figure 9 presents results for other countries and also for using Capital-Output ratios instead of Wealth-Income ratios. The left side panels show the Wealth-Income ratios, the right-side panels show Capital-Output ratios. The top panels show the US (so the top-left panel just repeats the top panel of Figure 8). The middle panels show Australia, Canada, and the UK. The bottom panels show France, Germany, Spain, and Sweden.

This negative correlation between the Wealth-Income ratio and the growth rate of real income we observe for the US since the mid-1980s is also present in most of the other countries, especially at over the post-1960 period (ie. after major effects of the world wars). The main exception is Spain where the correlation is essentially zero. Our result remains robust to using the capital-output ratio instead of the wealth-income ratio for the US (from Penn World Table data).

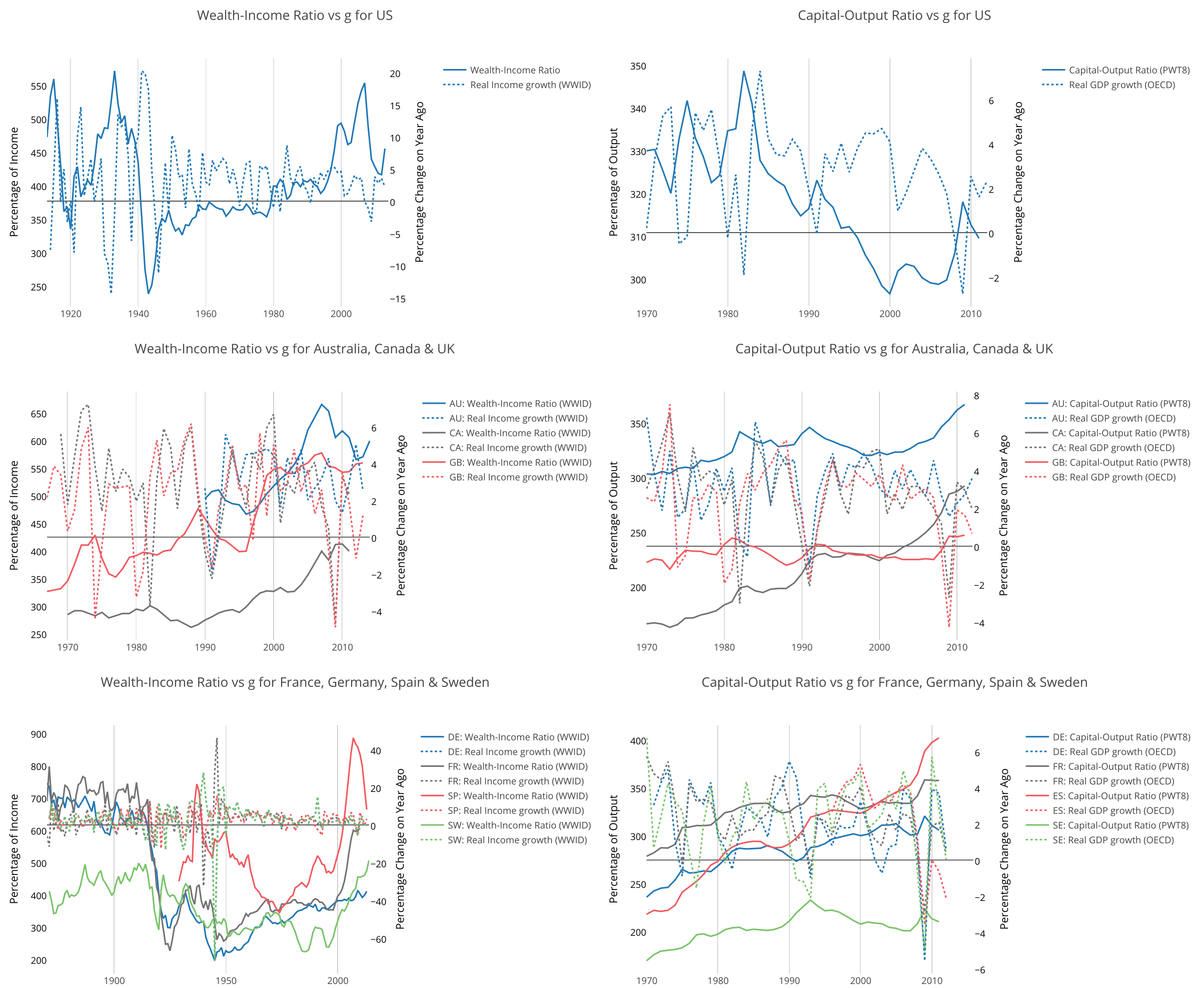
Table 3 presents a brief statistical summary of what we see in these graphs. The OLS coefficients on g from a regression of Wealth-Income ratios (or Capital-Output ratios) on a constant terms and g are given. We observe that all but one of the coefficients is negative, and that non-negative coefficient is very close to zero (both in absolute terms and in statistical significance). We observe that despite the small number of observations some of the coefficients from the Wealth-Income regressions, and most from the Capital-Output regressions, are statistically significant.[27] We conclude that more formal techniques confirm what we observed visually in the graphs: a negative correlation between Wealth-Income ratio and rate of income growth (g).

[27]We note that the Capital-Output ratios all cover just the post world-wars period.



* Data come from the WWID. The data shown are 'Net National Wealth as a percentage of National Income' and the growth rate of income calculated as the percentage change in 'National Income' since the previous year. As a comparison for the income growth data the OECD data on the growth rate of real GDP is included for the later part of the time period for which it is available. The second panel shows the same data, but smoothed using an 11-period centered moving average filter. The third papel shows the same data, but smoothed using a 5-lag moving average filter.

Figure 8: Negative Correlation between Wealth-Income ratio and rate of income growth (g) in US



* For left-side panels data come from the WWID. The data shown are 'Net National Wealth as a percentage of National Income' and the growth rate of income calculated as the percentage change in 'National Income' since the previous year. For right-side panels data come from the Penn World Table 8 (accessed via FRED) and show 'Capital Stock at Constant National Prices for *Country*' and 'Real GDP at Constant National Prices for *Country*'; index year for, eg., US is 2005.

Figure 9: Negative Correlation between Wealth-Income ratio and rate of income growth (g)

Table 3: Negative Correlation between Wealth-Income ratio and rate of income growth (g)
Country Correlation Coefficients between Wealth-Income and g

	OLS coefficient and (p-value)			
	Wealth-Income Ratio		Capital-Income Ratio	
Australia	-6.0061	(0.4313)	-2.8042	(0.0415)
Canada	-3.6113	(0.1568)	-6.2509	(0.0155)
France	-5.3643	(0.0066)	-7.3981	(0.0000)
Germany	-1.5521	(0.3170)	-2.0071	(0.2200)
Spain	0.1395	(0.9693)	-9.8835	(0.0000)
Sweden	-0.9348	(0.1719)	-1.9495	(0.0456)
UK	-2.7143	(0.5521)	-1.9374	(0.0001)
US	-2.4310	(0.0366)	-1.0525	(0.2871)

All data are observed annually. The number of observations varies across countries, and for each country reflects all available data. Correlation coefficients are calculated by OLS regression (with a constant term) and p-values are for the standard null hypothesis of H0: coefficient=0.

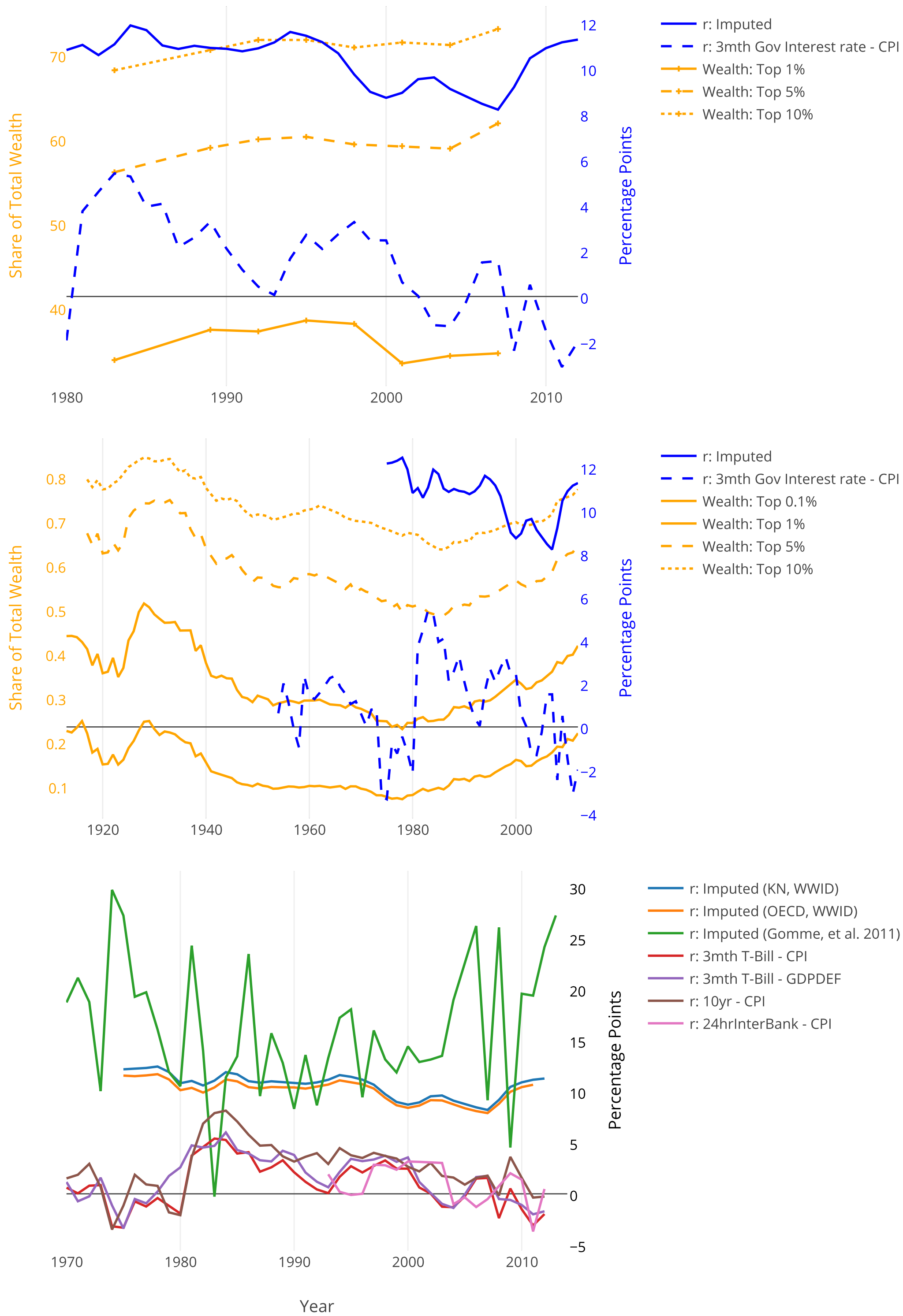
Correlation (ii): Negative Correlation between Wealth Inequality and rate of return to wealth (r):

Table 4: Negative Correlation between Wealth Inequality and rate of return to wealth (r)
Correlation Coefficients between Wealth Inequality and r (for US)

	Wealth Inequality Measure		
	Share of Top 10% (SCF)	Gini Coeff. (SCF)	Share of Top 10% (SZ2016)
	[OLS coefficient (p-value)]		
Rate of Return to Wealth (KN,WWID)	-0.5958 (0.1985)	-0.0038 (0.2988)	-1.0631 (0.0276)
Rate of Return to Wealth (OECD,WWID)	-0.5996 (0.2423)	-0.0038 (0.3489)	-1.2546 (0.0100)
Rate of Return to Capital (GommeEtAl)	0.1259 (0.1649)	0.0014 (0.0329)	0.1346 (0.0048)
Interest rate (IMF 3mth, OECD CPI)	-0.4262 (0.0899)	-0.0031 (0.1146)	-0.8261 (0.0000)
Interest rate (IMF 3mth, OECD GDPDEF)	-0.3481 (0.2023)	-0.0023 (0.2807)	-0.9320 (0.0000)
Interest rate (FedRes 10yr, BLS CPI)	-0.5908 (0.0090)	-0.0049 (0.0024)	-0.6606 (0.0002)
Interest rate (OECD 24hrIB, OECD CPI)	0.0048 (0.9835)	-0.0002 (0.8940)	-0.9413 (0.0035)

All regressions are run on the available annual data; note that SCF data is only periodic so contains small number of observations. The number of observations varies across and reflects all available data for each time series post-1954. Correlation coefficients are calculated by OLS regression (with a constant term) and p-values are for the standard null hypothesis of H0: coefficient=0. For purpose of regression the shares are measured as 0 to 100. First data observations are 1975 for KN data, 1970 for OECD data, 1954 otherwise.

The top panel of Figure 10 shows the negative correlation between Wealth Inequality and the rate of return to wealth (r) present in the US economy since the mid-1980s; a repeat of that in the body of this paper. The data on Wealth Inequality are the top percentile shares of wealth taken from the SCF. Since the SCF data only begin in the mid-1980s (1983) we begin the graph then. The middle panel shows that if we instead used Wealth Inequality measures of Saez and Zucman (2016), together with the longer time periods that this allows, we would continue to see the same negative



[*] Data on wealth inequality in top panel from the Survey of Consumer Finances (SCF), as summarized Table 2 of Wolff (2010). We show the Share of Total Wealth of the Top 1% (5 & 10) of Wealth. Data on wealth inequality in the second panel are tax-unit level data from Saez and Zucman (2016). In first two panels data on the rate of return to wealth (r) is measured in two ways. One, following the literature standard is as the difference between 3-month Treasury bills and the CPI inflation rate (in first panel data comes from IMF and OECD, in second panel from US Federal Reserve and Bureau of Labour Statistics). The other is as the ratio of the Capital Share of Output (rK/Y) divided by the Wealth-Income ratio (K/Y), using data from Karabarbounis and Neiman (2014) and WWID respectively. The third panel presents the same two concepts r, but using different data (eg. 10yr instead of 3mth bonds, or OECD data on labour share of income); all series not already used in earlier panels come from IMF and OECD (some of the data series in third panel do begin before 1970).

Figure 10: Negative Correlation between Wealth Inequality and rate of return to wealth (r)

correlation. Both the top and middle panels use the same two measures of the rate of return to wealth (as previously discussed). The first is the real interest rate measured as the difference between the 3-month interest rate on Treasury bills and the CPI inflation rate. As discussed in the introduction, while standard in both the literature on r-g and on secular stagnation, it is not clear that this is the appropriate measure. We therefore also consider a more direct measure of the return to wealth: observing that in a neoclassical growth model[28] the capital share of output is rK/Y, and the wealth-income ratio is K/Y, so we can measure the rate of return to wealth as r equals capital share of output divided by the wealth-income ratio. Our other measure does this using the capital share of output data of Karabarbounis and Neiman (2014)[29] and the wealth-income ratio data of the WWID. The third panel compares many alternative ways to calculate these two measures of the rate of return to wealth such as using interest rates on 10-year Treasury bonds, or 24-hour Interbank market rates; using the GDP deflator instead of CPI inflation; and using OECD data on the capital share of output. We also show the (pre-tax) rate of return to capital calculated by Gomme et al. (2011). As can be seen, with the exception of using the rate of return to capital, all of these other measures of wealth inequality and r presented in the second and third panels show the same trends, and so our finding of a negative correlation between Wealth Inequality and the rate of return to wealth (r) is robust to these alternative measures. This finding is confirmed by Table 4 which presents the correlation coefficients and their statistical significance for the various combinations of wealth inequality and rate of return to wealth measures; all are negative and those relating to the longer time series of Saez and Zucman (2016) are often statistically significant; with the exception of the rate of return to capital.

As previously discussed in relation to r-g and wealth inequality, we do not here consider robustness to the use of other countries due to the paucity of detailed wealth inequality data.

[28] Including that with heterogeneous agents and incomplete markets which is presented in this paper.
[29] Technically, 1 minus their labour share of output data.

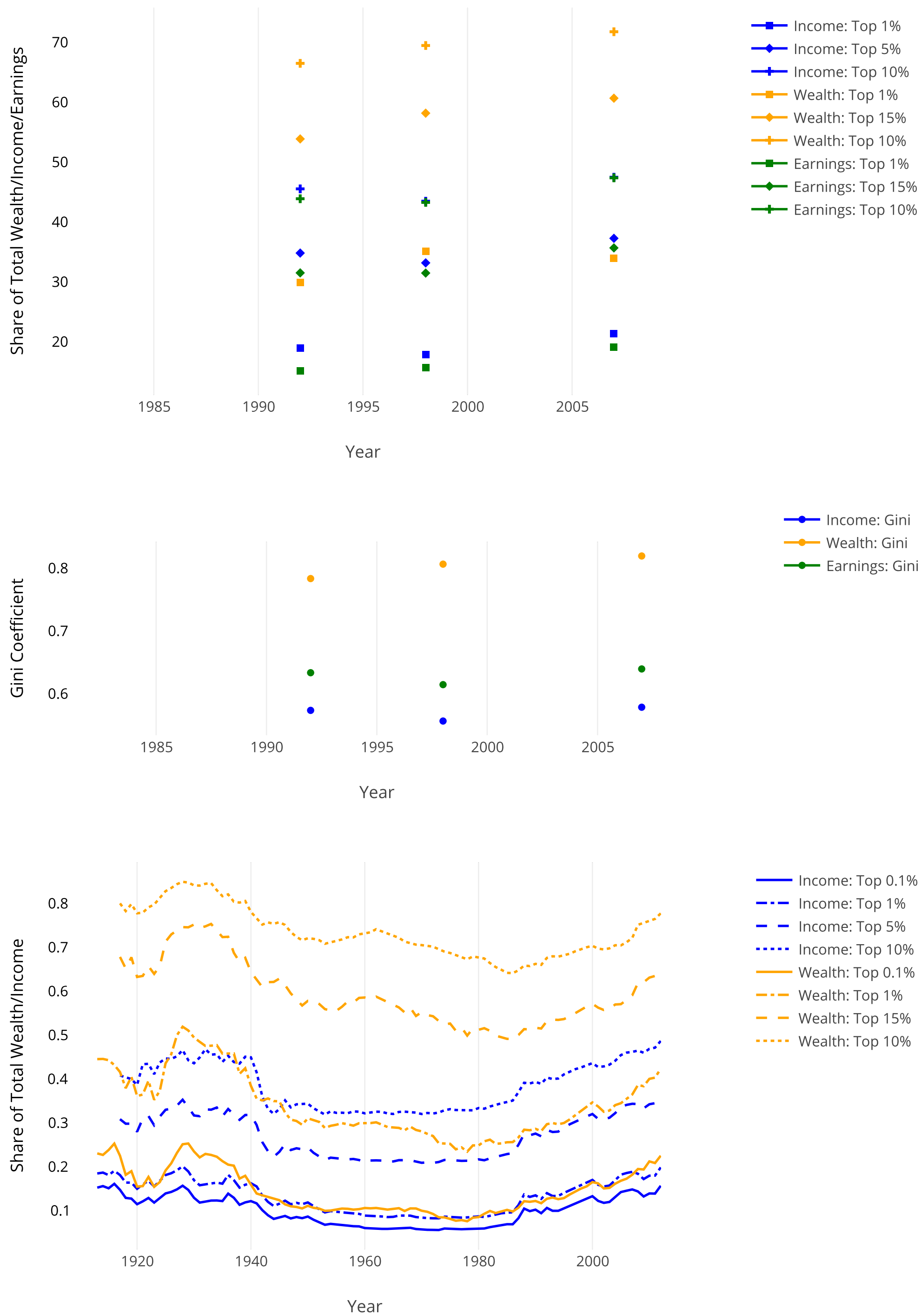
Correlation (iii): Positive Correlation between Income Inequality and Wealth Inequality:

We saw in Figure 4 the positive correlation between income inequality and wealth inequality observed in the US since the mid-1980s. It used household level data from the SCF; with the summary numbers are taken from Table 2 of Wolff (2010). A tighter link between data and model would require that we instead look at the correlation between Earnings (income from labour) Inequality and Wealth Inequality. The first and second panels of Figure 11 show for the limited summary SCF data available, (Díaz-Giménez et al., 1997; Burdia Rodriguez et al., 2002; Díaz-Giménez et al., 2011), that it would not change our conclusions if we had used Earnings Inequality instead of Income Inequality. As discussed in the introduction we prefer the SCF data since it is at the household level and the natural interpretation of infinite-lived agents (as in our model) is as dynastic households. For robustness the bottom panel of Figure 11 shows Income Inequality (from WWID) and Wealth inequality data (from Saez and Zucman (2016)) on Shares of Income/Wealth held by Top Percentiles. We see an even stronger positive correlation, and for a much longer time period.

We do not look at the correlation between income and wealth inequality in other countries. Partly because it seems very uncontroversial, but mainly just due to data limitations with regards to wealth inequality data, especially that for top percentile shares. Few countries have surveys like the SCF which deliberately oversample high income/wealth individuals to ensure that they are well measured, and numbers based on a capitalization of capital income like the Saez and Zucman (2016) numbers do not exist for other countries. A few countries do have good wealth inequality numbers (Roine and Waldenstrom, 2009), albeit often based on estate tax data (Kopczuk and Saez, 2004; Piketty et al., 2006), precursory investigation (not reported) suggests that our results are robust to these.

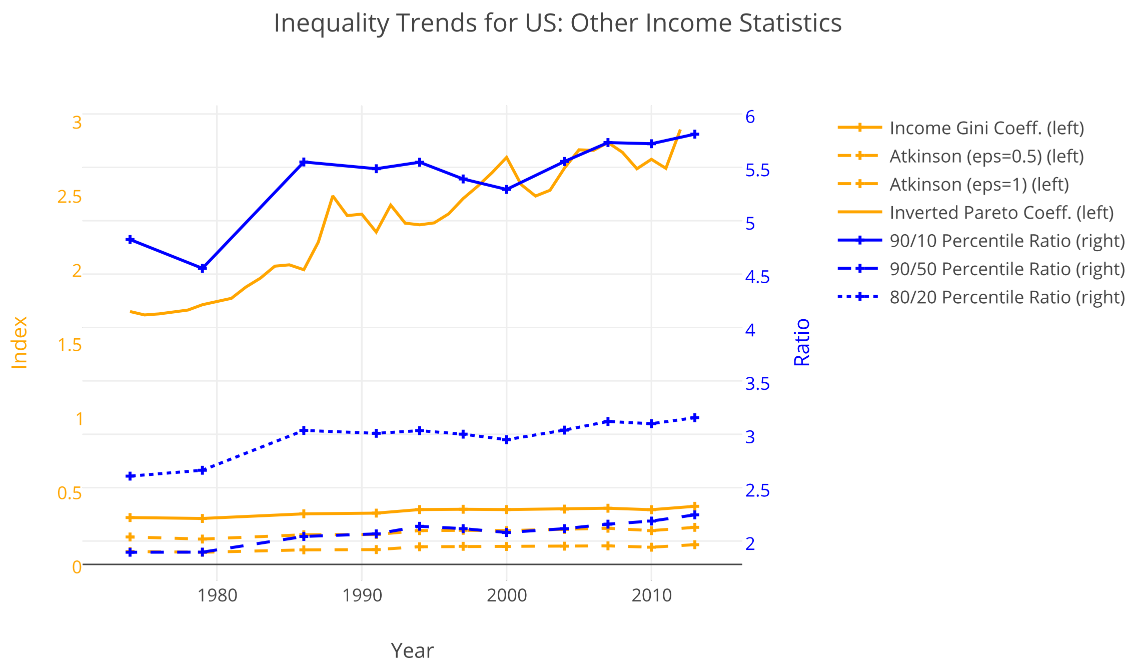
We do perform one further check, only possible for our income inequality data (not with the wealth inequality), and that is our choice of measure. Most of our figures look at, eg. the share of total income held by the top one-percent of incomes, as a measure of income inequality. Since our model is about inequality more generally and not just the top shares we want to be sure that a focus on these top shares when looking at the data is not misleading. For the US Figure 12 shows many other measures of income inequality such as Atkinson indices and ratios such as the income of the 90th income percentile to the 50th income percentile. These are taken from the Luxembourg Income Study, so observations are not annual. The figure also shows the Inverted Pareto Coefficient (from WWID) which is a commonly calculated statistic in some parts of the inequality literature. All these other measures clearly display the same upward trend over the period (since 1974) for which they are available as we saw in the top percentile shares. We conclude that our use of top percentile shares in the rest of our empirical work is not misleading.[30]

[30]We have not performed this test for wealth or earnings inequality, or for other countries.



* Data for first and second panels are from the Survey of Consumer Finances (SCF), as summarized in Díaz-Giménez et al. (1997); Burdia Rodriguez et al. (2002); Díaz-Giménez et al. (2011). The top panel show the Total Income (Wealth) shares of the Top 1% of Incomes (Wealth); note that while there is overlap these are not the same households. The second panel shows the Gini coefficients for income and wealth. The bottom panel presents tax-unit level measures of the top percentile shares with the income data taken from WWID and the wealth data from Saez and Zucman (2016).

Figure 11: Positive Correlation between Income Inequality and Wealth Inequality in the US.



* Data are from the Luxembourg Income Study, with exception of Inverted Pareto Coefficient which is from WWID. The Luxembourg Income Study is not annual, so markers indicate actual observations, the first available one of which is 1974.

Figure 12: Alternative Measures of Income Inequality in the US.

Correlation (iv): Positive Correlation between Income Inequality and Income Risk:

Figure Our US data on income risk measured as the ratio of the variance of income to the variance of consumption comes from the Consumer Expenditure Survey, as summarized in Heathcote, Perri, and Violante (2010). Fisher, Johnson, and Smeeding (2015) show that we get similar results for 1984-2011 if we use a wide demographic (rather than just the working-age white males used in the Heathcote, Perri, and Violante (2010) numbers).

Also, Moffitt and Gottschalk (2009, 2012) results suggest an increase in income risk.

B Some Quantitative Results from Models

Tables 5, 6 and 7 provide further simulation results for some standard parameter values that give an idea of the predicted size of the relationship for the model without endogenous labour.

Table 5: General Equilibrium Interest Rates in Aiyagari (1994)

A. Net Return to Capital in %/Aggregate savings rate in % ($\sigma = 0.2$)			
ρ/μ	1	3	5
0.0	4.1417/37.84	4.0669/27.23	3.9671/19.80
0.3	4.1168/18.37	4.0419/26.21	3.9172/23.79
0.6	4.1168/26.43	3.9421/23.72	3.7425/24.82
0.9	4.0170/25.42	3.5679/24.78	3.0439/26.20
B. Net Return to Capital in %/Aggregate savings rate in % ($\sigma = 0.4$)			
ρ/μ	1	3	5
0.0	4.0918/23.89	3.8673/24.42	3.5429/24.87
0.3	3.9920/23.81	3.5679/24.74	3.0689/26.03
0.6	3.8423/24.08	3.0689/26.03	2.2206/28.06
0.9	3.5928/24.89	2.0958/28.64	0.6737/33.30

Replication of Table 2 of Aiyagari (1994) using grid sizes $n_k = 512$, $n_z = 21$, $n_p = 251$

Table 6: Interest Rates and Inequality in Aiyagari (1994): Gini Coefficient
Gini Coefficients for Earnings, Income, and Wealth

A. Earnings Gini/Income Gini/Wealth Gini ($\sigma = 0.2$)			
ρ/μ	1	3	5
0.0	0.11/0.17/0.19	0.11/0.32/0.37	0.11/0.31/0.40
0.3	0.11/0.30/0.39	0.11/0.31/0.36	0.11/0.30/0.36
0.6	0.11/0.33/0.39	0.11/0.31/0.38	0.11/0.30/0.35
0.9	0.12/0.43/0.51	0.12/0.40/0.47	0.12/0.37/0.43
B. Earnings Gini/Income Gini/Wealth Gini ($\sigma = 0.4$)			
ρ/μ	1	3	5
0.0	0.22/0.30/0.36	0.22/0.29/0.34	0.22/0.27/0.32
0.3	0.22/0.33/0.39	0.22/0.30/0.35	0.22/0.27/0.32
0.6	0.22/0.36/0.43	0.22/0.32/0.38	0.22/0.29/0.34
0.9	0.23/0.48/0.55	0.23/0.43/0.48	0.23/0.38/0.42

Aiyagari (1994) reports a few Gini coefficients, but no Table.
Uses grid sizes $n_k = 512$, $n_z = 21$, $n_p = 251$

C Theory for Balanced Growth Path in inequality-decreases-r model

This appendix presents theory related to the balanced growth path in the inequality-decreases-r model. I will here maintain the assumption that Z is finite valued. It is not necessary that

Table 7: Interest Rates and Inequality in Aiyagari (1994): Inverted Pareto Coefficient (Inverse) Pareto Coefficients for Earnings, Income, and Wealth

A. Earnings Pareto Coeff/Income Pareto Coeff/Wealth Pareto Coeff ($\sigma = 0.2$)			
ρ/μ	1	3	5
0.0	0.65/0.44/0.41	0.53/0.44/0.42	0.50/0.53/0.52
0.3	0.54/0.62/0.60	0.53/0.48/0.47	0.51/0.51/0.50
0.6	0.59/0.53/0.52	0.58/0.60/0.56	0.59/0.57/0.56
0.9	0.60/0.50/0.49	0.58/0.50/0.48	0.56/0.47/0.46
B. Earnings Pareto Coeff/Income Pareto Coeff/Wealth Pareto Coeff ($\sigma = 0.4$)			
ρ/μ	1	3	5
0.0	0.50/0.52/0.51	0.51/0.54/0.50	0.52/0.52/0.51
0.3	0.53/0.54/0.52	0.52/0.52/0.49	0.52/0.50/0.48
0.6	0.55/0.52/0.51	0.54/0.50/0.49	0.53/0.47/0.46
0.9	0.60/0.54/0.52	0.60/0.52/0.52	0.61/0.52/0.52

Aiyagari (1994) does not report Inverted Pareto coefficients. Uses grid sizes $n_k = 512$, $n_z = 21$, $n_p = 251$

Z be finite valued, or even bounded; these assumptions are made for simplicity as they both ease the Markov theory required and provide a better fit with the computational methods used. This appendix shows that a Stationary Competitive Equilbrium in the growth-adjusted model is a Balanced Growth Path in the original model. This involves two main steps. The first step is to show the renormalized utility function (defined below) used in the growth-adjusted model is in fact a utility function, this is addressed in Propostion 1. The second step is to show that a stationary equilibrium of the growth-adjusted model is a balanced-growth path in the original model. The second step is based around the four conditions that make up each of the equilibrium definitions, namely that (i) the household optimizes, (ii) aggregates are determined by individual actions, (iii) markets clear, and (iv) a condition on the distribution of agents. For each of these four conditions I show that the renormalizations relating the variables in the growth-adjusted economy to the original economy imply that any Stationary (Competitive) Equilbrium of the growth-adjusted economy is also a (Competitive) Balanced-Growth-Path Equilibrium of the original economy, and vice-versa; Proposition 2.

I begin by repeating, for the readers convenience, the definition of the growth-adjustment normalization of equations (8)-(14).

$$\begin{aligned}
\hat{c}_t &\equiv c_t/E_t \\
\hat{\beta} &\equiv \beta(1+g)^{1-\gamma} \\
\hat{k}_t(z^{t-1}) &\equiv k_t(z^{t-1})/E_t \\
\underline{\hat{\mathrm{k}}}_t(z) &\equiv \underline{\hat{\mathrm{k}}}_t(z)/E_{t+1} \\
\hat{w}_t &\equiv w_t,\ \hat{r}_t \equiv r_t \\
\hat{K}_t &\equiv K_t/E_t,\ \hat{L}_t \equiv L_t/E_t \\
\hat{\mu}_t(\hat{k}, z) &\equiv \mu_t(k/E_t, z)
\end{aligned}$$

for $t = 0, 1, 2, ...$, scaled to recover an allocation in the original growth model by $E_t = (1+g)^t E_0$, $t = 0, 1, 2, ...$, where E_0 is initial aggregate labour productivity units.

One further piece of notation that will be useful in explaining and proving the required results is to define the utility function in the growth-adjusted normalization based on incorporating the normalizations to consumption and beta into the original utility function,

$$\hat{U}(\{\hat{c}_t(z^t)\}) \equiv \sum_{t=0}^{\infty} \int_{z^t \geq z_0} \hat{\beta}^t Q^t(z_0, dz^t) \frac{\hat{c}_t(z^t)^{1-\gamma}}{1-\gamma}$$

where $\hat{U}$ denotes the utility function in the growth-adjusted economy.

The following proposition covers step 1 — showing that the utility function redefined on the renormalized variables ($\hat{U}$ as defined immediately above) used in the growth-adjusted model implies the same preferences as that in the original economy — and is a subcase of Proposition 4.1 of Krueger and Lustig (2010).[31]

Proposition 1. *Households rank consumption share allocations in the growth-adjusted model in the same way as they rank the corresponding consumption allocations in the original model with growth: for any (z^t, E_t) and any two consumption allocations $\{c_t(z^t)\}$, $\{c'_t(z^t)\}$*

$$U(\{c_t(z^t)\})(z^t, E_t) \geq U(\{c'_t(z^t)\})(z^t, E_t) \iff \hat{U}(\{\hat{c}_t(z^t)\})(z^t, E_t) \geq \hat{U}(\{\hat{c}'_t(z^t)\})(z^t, E_t)$$

where the transformation of consumption into consumption shares is given by equation (8).

Proof. By definition the continuation utility of an agent from consumption sequence $\{c_t(z^t)\}$ starting at history (z^t, E_t) is given by the recursion,

$$U(\{c_t(z^t)\}; E_t) = \frac{c_t(z^t)^{1-\gamma}}{1-\gamma} + \beta \int_{z_{t+1}} U(\{c_{t+1}(z^t, z_{t+1})\}; E_{t+1}) Qt(z_t, dz_{t+1})$$

[31] Krueger and Lustig (2010) develop this proposition for the purpose of studying asset pricing and not balanced-growth paths, but it turns out to be equally useful here.

Dividing both sides by $E_t^{1-\gamma}$ gives

$$\frac{U(\{c_t(z^t)\};E_t)}{E_t^{1-\gamma}} = \frac{\hat{c}_t(z^t)^{1-\gamma}}{1-\gamma} + \beta \int_{z_{t+1}} \frac{U(\{c_{t+1}(z^t,z_{t+1})\};E_{t+1})}{E_{t+1}} \frac{E_{t+1}}{E_t} Qt(z_t, dz_{t+1})$$

Define a new continuation utility index $\hat{U}(\cdot)$ as follows

$$\hat{U}(\{\hat{c}_t(z^t)\}) \equiv \frac{U(\{c_t(z^t)\};E_t)}{E_t^{1-\gamma}}$$

It follows that

$$\hat{U}(\{\hat{c}_t(z^t)\}) \frac{\hat{c}_t(z^t)^{1-\gamma}}{1-\gamma} + \hat{\beta} \int_{z_{t+1}} \hat{U}(\{\hat{c}_{t+1}(z^t,z_{t+1})\}) Qt(z_t, dz_{t+1})$$

Thus for two consumption sequences it follows that U orders them in exactly the same manner as $\hat{U}$ orders the corresponding growth-adjusted consumption sequences. *Q.E.D.*

The following proposition covers step 2 —

Proposition 2. *Utility function $\hat{U}$, allocation decisions $\{\hat{c}_t(z^t), \hat{k}_t(z^t)\}_{t=0}^{\infty}$, prices $\{\hat{w}, \hat{r}\}$, aggregate capital and labour $\{\hat{K}, \hat{L}\}$, and a measure of agents $\{\mu(k,z)\}$ are a stationary competitive equilibrium of the growth-adjusted economy if and only if Utility function U, allocation decisions $\{c_t(z^t), k_t(z^t)\}_{t=0}^{\infty}$, prices $\{w, r\}$, aggregate capital and labour $\{K_t, L_t\}_{t=0}^{\infty}$, and a measure of agents $\{\mu_t(k,z)\}_{t=0}^{\infty}$ are a balanced-growth path of the original economy. Where the relationship between the growth-adjusted economy variables and those of the original economy are given by equations (8)-(14).*

Proof. The proof proceeds in four parts, corresponding to the four components of the equilibrium definitions: (i) the household optimizes, (ii) aggregates are determined by individual actions, (iii) markets clear, and (iv) a condition on the distribution of agents. In all four part the proof involves proving that an equilibrium solution for one definition also satisfies that condition for the other definition via the renormalization.

Part (i) - the household optimizes: since the return functions are strictly concave and the choice sets are strictly convex the combination of the first order conditions, the slackness conditions, and the transversality conditions together are both necessary and sufficient conditions for household optimization. In what follows I go one-by-one through these conditions from the households problem in the original economy and show that substitution based on the renormalization defining the growth-adjusted economy gives the optimality conditions for the households problem in the growth-adjusted economy; these substitions could equally be reversed. I begin with the first order conditions of the household problem from the original economy (imposing constant r and w conditions of the

BGP), these are given by,

$$\beta^t \Pr(z^t)u'(c_t(z^t)) = \beta^t Pr(z^t)\lambda_t(z^t)(-1)$$
$$0 = \beta^t \Pr(z^t)\lambda_t(z^t)(-1) + \sum_{z_{t+1}}[\beta^{t+1}Pr(z^{t+1})\lambda_{t+1}(z^t, z_{t+1})(1+r)] + \beta^t \Pr(z^t)\kappa_t(z^t)(-1)$$
$$\underline{\mathrm{k}}(E_t) - k_{t+1}(z^t) \leq 0$$
$$c_t(z^t) + k_{t+1}(z^t) = wz_tE_t + (1+r)k_t(z^{t-1})$$
$$\kappa_t(z^t) \geq 0$$

for all z^t, for all t; where $\beta^t Pr(z^t)\lambda_t(z^t)$ are the lagrange multipliers associated with the budget constraints, equation (3), and $\beta^t Pr(z^t)\kappa_t(z^t)$ are the lagrange multipliers associated with the borrowing constraints, equation (4). Define $\hat{\lambda}_t(z^t) = E_t^{\gamma}\lambda_t(z^t)$ and $\hat{\kappa}_t(z^t) = (1+g)^{1-\gamma}E_{t+1}^{\gamma}\kappa_t(z^t)$. Then by substituting all variables for their hatted equivalents using (8)-(14) and the definitions of $\hat{\lambda}_t(z^t)$ and $\hat{\kappa}_t(z^t)$ just given, we get

$$\hat{\beta}^t \Pr(z^t)u'(\hat{c}_t(z^t)) = \beta^t Pr(z^t)\hat{\lambda}_t(z^t)(-1)$$
$$0 = \beta^t \Pr(z^t)\hat{\lambda}_t(z^t)(-(1+g)) + \sum_{z_{t+1}}[\hat{\beta}^{t+1}Pr(z^{t+1})\hat{\lambda}_{t+1}(z^t, z_{t+1})(1+r)] + \hat{\beta}^t \Pr(z^t)\hat{\kappa}_t(z^t)(-1)$$
$$\hat{\underline{\mathrm{k}}} - \hat{k}_{t+1}(z^t) \leq 0$$
$$\hat{c}_t(z^t) + (1+g)\hat{k}_{t+1}(z^t) = \hat{w}z_t + (1+\hat{r})\hat{k}_t(z^{t-1})$$
$$\hat{\kappa}_t(z^t) \geq 0$$

for all z^t, for all t. Which are the first order conditions of the households problem in the growth-adjusted economy. Where $\hat{\beta}^t Pr(z^t)\hat{\lambda}_t(z^t)$ are the lagrange multipliers associated with the budget constraints, equation (16), and $\hat{\beta}^t Pr(z^t)\hat{\kappa}_t(z^t)$ are the lagrange multipliers associated with the borrowing constraints, equation (17).

The slackness conditions for the original economy are given by

$$\kappa_t(z^t)[\underline{\mathrm{k}}(E_t) - k_{t+1}(z^t)] = 0$$

for all z^t, for all t. By substitution of variables for their growth-adjusted (hatted) equivalents, and canceling some variables we know to be non-zero, we get

$$\hat{\kappa}_t(z^t)[\hat{\underline{\mathrm{k}}} - \hat{k}_{t+1}(z^t)] = 0$$

for all z^t, for all t. Which are the slackness conditions for the households problem in the growth-adjusted economy.

The transversality condition for the original economy is given by

$$\lim_{T\to\infty} \beta^T u'(c_T(z^T))k_{T+1}(z^T) = 0$$

By substitution of variables for their growth-adjusted (hatted) equivalents we get

$$\lim_{T\to\infty} \hat{\beta}^T u'(\hat{c}_T(z^T))\hat{k}_{T+1}(z^T) = 0$$

Which is the slackness condition for the households problem in the growth-adjusted economy. Thus the renormalized solution of the households problem in the original economy solves the households problem in the growth-adjusted economy; and vice-versa.

Part (ii) - aggregates are determined by individual actions: The aggregation conditions for the original economy are given by $K_t = \int k_t(z^t) d\mu_t(k_t(z^t), z^t)$, and $L_t = \int z^t E_t d\mu_t(k_t(z^t), z^t)$. Simply substituting the variables for their hatted equivalents, and taking advantage of the property imposed on the distribution function μ_t as part of the definition of a Balanced-Growth Path we get the aggregation conditions of a stationary equilibrium of the growth-adjusted economy, namely $\hat{K} = \int \hat{k}_t(z^t) d\hat{\mu}(\hat{k}_t(z^t), z^t)$, and $\hat{L} = \int z^t d\hat{\mu}(\hat{k}_t(z^t), z^t)$. And vice-versa.

Part (iii) - markets clear: The market clearance for a Balanced Growth Path in the original economy are given by $r - (\alpha K_t^{\alpha-1} L_t^{1-\alpha} - \delta) = 0$ and $w - (1-\alpha) K_t^{\alpha} L_t^{-\alpha} = 0$ for all t. Simply substituting the variables for their hatted equivalents we get the market clearance conditions of a stationary equilibrium of the growth-adjusted economy, namely $\hat{r} - (\alpha \hat{K}^{\alpha-1} \hat{L}^{1-\alpha} - \delta) = 0$ and $\hat{w} - (1-\alpha) \hat{K}^{\alpha} \hat{L}^{-\alpha} = 0$

Part (iv) - a condition on the distribution of agents: The condition on the distribution of agents for a Balanced Growth Path in the original economy is that the measure of agents is invariant on a renormalization of the state-space by a constant function of aggregate endowment, namely

$$\mu(k_t(z^t)/E_t, z^t) = \int \int_{z^t \geq z^{t-1}} (k_t(z^{t-1})/E_t, z^{t-1}) \mu(k^{t-1}/E_{t-1}, z^{t-1}) Q(z_{t-1}, dz_t)$$

Simply substituting the variables for their hatted equivalents we get an invariant measure that satisfies the condition on the distribution of agents for a stationary distribution in the growth-adjusted economy, namely

$$\hat{\mu}(\hat{k}_t(z^t), z^t) = \int \int_{z^t \geq z^{t-1}} (\hat{k}_t(z^{t-1}), z^{t-1}) \hat{\mu}(\hat{k}^{t-1}, z^{t-1}) Q(z_{t-1}, dz_t)$$

Q.E.D.

Since computation of the model stationary equilibrium of the growth-adjusted model is based on the models recursive forumulation, the definition of Stationary (Recursive) Competitive Equilibrium (Definition 2) is now given in recursive value function iteration notation,

Definition 3. *A Stationary (Recursive) Competitive Equilbrium is an agents value function $V(k, z)$; agents policy function $k' = g(k, z)$; an interest rate r and wage w; aggregate capital K and labour H; and a measure of agents $\mu(k, z)$; such that*

1. *Given prices r & w, the agents value function $V(k, z)$ and policy function $k' = g(k, z)$ solve*

the agents problem

$$V(k,z) = \max_{k'} \left\{ u(c) + \beta \int V(k',z') Q(z,dz') \right\}$$
$$s.t. \quad c + k' = wz + (1+r)k$$
$$c \geq 0,\ k' \geq \underline{k}$$

2. *Aggregates are determined by individual actions:* $K = \int k d\mu(k,z)$, *and* $H = \int z d\mu(k,z)$

3. *Markets clear (in terms of prices):* $r - (\alpha K^{\alpha-1} H^{1-\alpha} - \delta) = 0$.

4. *The measure of agents is invariant,*

$$\mu(k,z) = \int \int \left[\int 1_{k=k'(\hat{k},z)} \mu(\hat{k},z) Q(z,dz') \right] d\hat{k} dz \tag{19}$$

where z is the labour supply shock which takes values in $Z = \{z_1, ..., z_{n_z}\}$ and evolves according to the *stationary* Markov transition function $Q(z,z')$.[32] Note that the wage is residualy determined by r.[33] The market clearance condition is more commonly expressed as $r = \alpha K^{\alpha-1} H^{1-\alpha} - \delta$, that the interest equals the marginal product of capital (minus the depreciation rate). Since $H = E(z) = 1$ the Cobb-Douglas production function is really only based on aggregate capital (in the sense that H is a fixed constant).

D Extension of Neoclassical Growth model with Heterogeneous Agents to include Endogenous Labour Supply

In this appendix I provide a full extension of the Neoclassical Growth model with Heterogeneous Agents model to the case of endogenous labour supply. This involves three main parts. First the extension of Section 3.1 covering the setup and equilibrium definitions. Second the extension of the results of Appendix C relating the balanced growth path to a stationary equilibrium of the renormalized model. Third the extension of the results of Section 3.2 relating inequality, the interest rate, the growth rate, and income risk in these models. I begin with the setup and equilibrium definitions.

D.1 Model Environment and Equilibrium Definitions

The model is a one-sector neoclassical growth model with a continuum of households facing idiosyncratic but no aggregate shocks, with incomplete markets (and borrowing constraints), who

[32]It is not necessary that Z be finite valued, or even bounded, but this simplifies the Markov theory required.
[33]The wage, which is given by the derivative of the Cobb-Douglas production with respect to labour, can be rewritten as a function of the interest rate and the parameters of the production function.

make decisions about consumption, leisure (labour supply), and savings.[34] The model is an extension of Huggett (1997) to endogenous labour supply, and when focusing on the balanced growth path it simplifies to a variation of Pijoan-Mas (2006). We begin by defining the model economy and giving the definition of Competitive Equilibrium and of a Balanced Growth Path. A growth-adjusted economy is then defined, a Stationary (Competitive) Equilibrium of which provides a (renormalized) Balanced Growth Path of the original economy. Section 3.2 provides results on the relationship between inequality and 'r-g' in the growth-adjusted economy, which are then be linked to the growth model via the results of this Section.

Model Economy with Endogenous Labour Supply: In the model infinitely lived households face a combination of an idiosyncratic stochastic process on labour efficiency units, z_t, and a deterministic growth of labour efficiency units, E_t; the idiosyncratic shock and deterministic aggregate growth interact multiplicitively. Denote a history of idiosyncratic shocks by $z^t = \{z_t, ..., z_1, z_0\}$. The households then make decisions on consumption, leisure (labour supply) and savings; given an interest rate path, $\{r_t\}$ and an wage path $\{w_t\}$.

The households order sequences of consumption and leisure $\{c_t(z^t, t), l_t(z^t, t)\}$ (and implicitly capital holdings $\{k_{t+1}(z^t, t)\}$) according to the utility function

$$U(\{c_t(z^t), l_t(z^t, t)\}) \equiv \sum_{t=0}^{\infty} \int_{z^t \geq z_0} \beta^t Q^t(z_0, dz^t) \frac{(c_t(z^t)^{1-\theta} l_t(z^t, t)^{\theta})^{1-\gamma}}{1-\gamma} \tag{20}$$

where $1/\gamma$ is the intertemporal elasticity of substition and θ is the share parameter for leisure in the composite commodity. The dependence of the utility function U on initial conditions, E_0 and an initial agent distribution, has been supressed in the notation as it will play no role. The idiosyncratic labour efficiency shock z_t is assumed to follow a stationary 1st-order Markov process with transition function $Q(z, dz')$. $Q^t(z^0, dz^t)$ is the t-step transition function defined iteratively by $Q^t(z_0, dz^t) = \int_{z^{t-1} \geq z_0} Q(z_{t-1}, dz_t) Q^{t-1}(z_0, dz^{t-1})$, from the stationary 1st-order Markov transtion function $Q(z, dz')$.

The households problem is to maximize utility subject to a budget constraint and a borrowing constraint,

$$\max_{\{c_t(z^t), l_t(z^t), k_t(z^t)\}} U(\{c_t(z^t), l_t(z^t)\}) \tag{21}$$

$$s.t. \quad c_t(z^t) + k_{t+1}(z^t) = w_t z^t E_t (1 - l_t(z^t)) + (1 + r_t) k_t(z^{t-1}), \quad \forall z^t, \forall t \tag{22}$$

$$k_{t+1}(z^t) \geq \underline{\mathrm{k}}(E_t), \quad \forall z^t, \forall t \tag{23}$$

$$k_1 \text{ given} \tag{24}$$

The economic growth of the economy is driven by E_t which grows deterministically according to $E_{t+1} = (1 + g) E_t$.

[34] Since the theoretical results, are based on F being increasing we will use leisure as the endogenous decision variable, rather than labour supply.

Individual household capital holdings and labour supply, given an interest rate and wage, aggregate to give aggregate capital holdings and aggregate labour supply (in efficiency units). The market clearance condition is that the interest rate and wage will be determined by perfect competition in the goods and labour markets together with a representative firm with Cobb-Douglas production function and a constant rate of labour-augmenting technological growth. All the different types of equilbrium definition considered below are general equilibria definitions for this model, and all involve perfect competition. I start with the definition of a Competitive Equilbrium, then a Balanced Growth Path equilibrium, and lastly a Stationary equilibrium.

Definition 4. *A Competitive Equilbrium is an agents utility function U and allocation decisions $\{c_t(z^t), l_t(z^t), k_{t+1}(z^t)\}_{t=0}^{\infty}$; sequences for interest rate and wage $\{r_t, w_t\}_{t=0}^{\infty}$; aggregate capital and labour $\{K_t, L_t\}_{t=0}^{\infty}$; and a sequence of measures of agents $\{\mu_t(k_t(z^t), z^t)\}_{t=0}^{\infty}$; such that*

1. *Given price sequences $\{r_t, w_t\}_{t=0}^{\infty}$, the agents utility function and allocation decisions solve the agents problem given by (20)-(24)*

2. *Aggregates are determined by individual actions: $K_t = \int k_t(z^t) d\mu_t(k_t(z^t), z^t)$, and $L_t = \int z^t E_t(1 - l_t(z^t)) d\mu(k_t(z^t), z^t)$*

3. *Markets clear (in terms of prices): $r_t - (\alpha K_t^{\alpha-1} L_t^{1-\alpha} - \delta) = 0$.*

4. *The measure of agents evolves according to:*

$$\mu_t(k_t(z^t), z^t) = \int \int_{z^t \geq z^{t-1}} (k_t(z^{t-1}), z^{t-1}) \mu_{t-1}(k^{t-1}(z^{t-2}), z^{t-1}) Q(z_{t-1}, dz_t) \tag{25}$$

where z is the labour supply shock which evolves according to Markov transition function $Q(z, dz')$.

In an economy such as this with heterogeneous agents a Balanced Growth Path is defined based on the concept that prices are constant and that the distribution of agents is invariant if considered the on a renormalization of agents state-space by a fixed function of the aggregate state.

Definition 5. *A Balanced Growth Path is an agents utility function U and allocation decisions $\{c_t(z_t), l_t(z_t), k_{t+1}(z_t)\}_{t=0}^{\infty}$; interest rate and wage $\{r, w\}$; aggregate capital and labour $\{K_t, L_t\}_{t=0}^{\infty}$; and a measure of agents $\{\mu(k, z)\}_{t=0}^{\infty}$; such that*

1. *Given prices $\{r, w\}$, the agents utility function and allocation decisions solve the agents problem given by (20)-(24)*

2. *Aggregates are determined by individual actions: $K_t = \int k_t(z^t) d\mu_t(k_t(z^t), z^t)$, and $L_t = \int z^t E_t d\mu(k_t(z^t), z^t)$*

3. *Markets clear (in terms of prices): $r_t - (\alpha K_t^{\alpha-1} L_t^{1-\alpha} - \delta) = 0$.*

4. *The measure of agents is generated by individuals allocation decisions and is invariant on a renormalized of the state-space by a constant function of aggregate endowment:*

$$\mu(k_t(z^t)/E_t, z^t) = \int \int_{z^t \geq z^{t-1}} (k_t(z^{t-1})/E_t, z^{t-1})\mu(k^{t-1}/E_{t-1}, z^{t-1})Q(z_{t-1}, dz_t) \qquad (26)$$

for some function f_E.

where z is the labour supply shock which evolves according to Markov transition function $Q(z, dz')$. Notice that μ is now time-invariant when considered on the space $(k^t/E_t, z^t)$, rather than (k^t, z^t).

Lastly the definition of a Stationary (Competitive) equilibrium is given,

Definition 6. *A Stationary Equilbrium is an agents utility function U and allocation decisions $\{c_t(z^t), l_t(z^t), k_{t+1}(z^t)\}_{t=0}^{\infty}$; sequences for interest rate and wage $\{r, w\}_{t=0}^{\infty}$; aggregate capital and labour $\{K, L\}_{t=0}^{\infty}$; and a measure of agents $\{\mu(k, z)\}_{t=0}^{\infty}$; such that*

1. *Given prices $\{r, w\}_{t=0}^{\infty}$, the agents utility function and allocation decisions solve the agents problem given by (20)-(24)*

2. *Aggregates are determined by individual actions: $K = \int k_t(z^t)d\mu(k_t(z^t), z^t)$, and $L = \int z^t E_t(1-l_t(z^t))d\mu(k_t(z^t), z^t)$*

3. *Markets clear (in terms of prices): $r - (\alpha K^{\alpha-1}L^{1-\alpha} - \delta) = 0$.*

4. *The measure of agents on the state-space renormalized by a constant function of aggregate endowment is invariant:*

$$\mu(k_t(z^t), z^t) = \int \int_{z^t \geq z^{t-1}} (k_t(z^{t-1}), z^{t-1})\mu(k^{t-1}, z^{t-1})Q(z_{t-1}, dz_t) \qquad (27)$$

where z is the labour supply shock which evolves according to Markov transition function $Q(z, dz')$. Notice that μ is now time-invariant and no renormalization is performed. The growth model economy with endogenous labour supply clearly does not have a Stationary equilibrium itself, but the concept will be useful in a renormalized version of the model.

As shown in section D.2 there is mapping between Balanced Growth Paths of the original model and stationary competitive equilibria of a growth-adjusted economy. This mapping is based on the definition of the growth-adjusted economy in terms of the hatted variables given by

$$\hat{c}_t \equiv c_t/E_t \tag{28}$$

$$\hat{l}_t \equiv l_t \tag{29}$$

$$\hat{\beta} \equiv \beta(1+g)^{(1-\gamma)(1-\theta)} \tag{30}$$

$$\hat{k}_t(z^{t-1}) \equiv k_t(z^{t-1})/E_t \tag{31}$$

$$\hat{\underline{\mathrm{k}}}_t(z) \equiv \underline{\mathrm{k}}_t(z)/E_t \tag{32}$$

$$\hat{w}_t \equiv w_t,\ \hat{r}_t \equiv r_t \tag{33}$$

$$\hat{K}_t \equiv K_t/E_t,\ \hat{L}_t \equiv L_t/E_t \tag{34}$$

$$\hat{\mu}_t(\hat{k}, z) \equiv \mu_t(k/E_t, z) \tag{35}$$

for $t = 0, 1, 2, ...$, scaled to recover an allocation in the original growth model by $E_t = (1+g)^t E_0$, $t = 0, 1, 2, ...$, where E_0 is initial aggregate labour productivity units. In the growth-adjusted economy the households decision problem, originally equations (20)-(24), is given by

$$\hat{U}(\{\hat{c}_t(z^t), \hat{l}_t(z^t)\}) \equiv \sum_{t=0}^{\infty} \int_{z^t \geq z_0} \hat{\beta}^t Q^t(z_0, dz^t) \frac{(\hat{c}_t(z^t)^{1-\theta} \hat{l}_t(z^t)^{\theta})^{1-\gamma}}{1-\gamma}$$

together with

$$\max_{\{\hat{c}_t(z^t), \{\hat{l}_t(z^t), \hat{k}_t(z^t)\}} \quad \hat{U}(\{\hat{c}_t(z^t), \hat{l}_t(z^t)\}) \tag{36}$$

$$s.t. \quad \hat{c}_t(z^t) + (1+g)\hat{k}_{t+1}(z^t) = \hat{w}_t z^t (1 - \hat{l}_t(z^t)) + (1+\hat{r}_t)\hat{k}_t(z^{t-1}), \quad \forall z^t, \forall t \tag{37}$$

$$\hat{k}_{t+1}(z^t) \geq \hat{\underline{\mathrm{k}}}, \quad \forall z^t, \forall t \tag{38}$$

$$\hat{k}_1 \text{ given} \tag{39}$$

In what follows I analyse the relationship between between interest rates, growth rates and inequality on a Balanced Growth Path by looking at the growth-adjusted economy. Importantly notice that the transformations of c, l, k, asset income, and wage earnings are such that standard inequality measures like Lorenz curves, Gini coefficients, and the shares of quintiles or percentiles will be exactly equal on the Balanced Growth Path to their values in the growth-adjusted economy; likewise for the measures of income risk.

D.2 Theory for Balanced Growth Path for model with endogenous labour

This subsection presents results for the model with endogenous labour supply. The structure exactly follows that for the exogeneous labour model as presented in Appendix C.

This appendix presents theory related to the balanced growth path in the inequality-decreases-r model. I will here maintain the assumption that Z is finite valued. It is not necessary that

Z be finite valued, or even bounded; these assumptions are made for simplicity as they both ease the Markov theory required and provide a better fit with the computational methods used. This appendix shows that a Stationary Competitive Equilbrium in the growth-adjusted model is a Balanced Growth Path in the original model. This involves two main steps. The first step is to show the renormalized utility function (defined below) used in the growth-adjusted model is in fact a utility function, this is addressed in Propostion 3. The second step is to show that a stationary equilibrium of the growth-adjusted model is a balanced-growth path in the original model. The second step is based around the four conditions that make up each of the equilibrium definitions, namely that (i) the household optimizes, (ii) aggregates are determined by individual actions, (iii) markets clear, and (iv) a condition on the distribution of agents. For each of these four conditions I show that the renormalizations relating the variables in the growth-adjusted economy to the original economy imply that any Stationary (Competitive) Equilbrium of the growth-adjusted economy is also a (Competitive) Balanced-Growth-Path Equilibrium of the original economy, and vice-versa; Proposition 4.

One further piece of notation that will be useful in explaining and proving the required results is to define the utility function in the growth-adjusted normalization based on incorporating the normalizations to consumption and beta into the original utility function,

$$\hat{U}(\{\hat{c}_t(z^t), \hat{l}_t(z^t)\}) \equiv \sum_{t=0}^{\infty} \int_{z^t \geq z_0} \hat{\beta}^t Q^t(z_0, dz^t) \frac{(\hat{c}_t(z^t)^{1-\theta} \hat{l}_t(z^t)^{\theta})^{1-\gamma}}{1-\gamma}$$

where $\hat{U}$ denotes the utility function in the growth-adjusted economy.

The following proposition covers step 1 — showing that the utility function redefined on the renormalized variables ($\hat{U}$ as defined immediately above) used in the growth-adjusted model implies the same preferences as that in the original economy — and is an extension of Proposition 1 to the case of endogenous labour supply.

Proposition 3. *Households rank consumption share allocations in the growth-adjusted model in the same way as they rank the corresponding consumption allocations in the original model with growth: for any* (z^t, E_t) *and any two consumption-leisure allocations* $\{c_t(z^t), l_t(z^t)\}$, $\{c'_t(z^t), l'_t(z^t)\}$

$$U(\{c_t(z^t), l_t(z^t)\})(z^t, E_t) \geq U(\{c'_t(z^t), l'_t(z^t)\})(z^t, E_t) \iff \hat{U}(\{\hat{c}_t(z^t), \hat{l}_t(z^t)\})(z^t, E_t) \geq \hat{U}(\{\hat{c}'_t(z^t), \hat{l}'_t(z^t)\})(z^t, E_t)$$

where the transformation of consumption into consumption shares is given by equation (28), and the transformation of leisure is given by equation (29).

Proof. By definition the continuation utility of an agent from consumption sequence $\{c_t(z^t)\}$ starting at history (z^t, E_t) is given by the recursion,

$$U(\{c_t(z^t), l_t(z^t)\}; E_t) = \frac{(c_t(z^t)^{1-\theta} l_t(z^t)^{\theta})^{1-\gamma}}{1-\gamma} + \beta \int_{z_{t+1}} U(\{c_{t+1}(z^t, z_{t+1}), l_{t+1}(z^t, z_{t+1})\}; E_{t+1}) Qt(z_t, dz_{t+1})$$

Dividing both sides by $E_t^{1-\gamma}$ gives

$$\frac{U(\{c_t(z^t)\}, l_t(z^t)\}; E_t)}{E_t^{1-\gamma}} = \frac{(\hat{c}_t(z^t)^{1-\theta}\hat{l}_t(z^t)^{\theta})^{1-\gamma}}{1-\gamma} + \beta \int_{z_{t+1}} \frac{U(\{c_{t+1}(z^t, z_{t+1}), l_{t+1}(z^t, z_{t+1})\}; E_{t+1})}{E_{t+1}} \frac{E_{t+1}}{E_t} Qt(z_t, dz_{t+1})$$

Define a new continuation utility index $\hat{U}(\cdot)$ as follows

$$\hat{U}(\{\hat{c}_t(z^t), \hat{l}_t(z^t)\}) \equiv \frac{U(\{c_t(z^t), l_t(z^t)\}; E_t)}{E_t^{1-\gamma}}$$

It follows that

$$\hat{U}(\{\hat{c}_t(z^t), \hat{l}_t(z^t)\}) \frac{(\hat{c}_t(z^t)^{1-\theta}\hat{l}_t(z^t)^{\theta})^{1-\gamma}}{1-\gamma} + \hat{\beta} \int_{z_{t+1}} \hat{U}(\{\hat{c}_{t+1}(z^t, z_{t+1}), \hat{l}_{t+1}(z^t, z_{t+1})\}) Qt(z_t, dz_{t+1})$$

Thus for two consumption sequences it follows that U orders them in exactly the same manner as $\hat{U}$ orders the corresponding growth-adjusted consumption sequences. *Q.E.D.*

The following proposition covers step 2 —

Proposition 4. *Utility function $\hat{U}$, allocation decisions $\{\hat{c}_t(z^t), \hat{l}_t(z^t), \hat{k}_t(z^t)\}_{t=0}^{\infty}$, prices $\{\hat{w}, \hat{r}\}$, aggregate capital and labour $\{\hat{K}, \hat{L}\}$, and a measure of agents $\{\mu(k, z)\}$ are a stationary competitive equilibrium of the growth-adjusted economy if and only if Utility function U, allocation decisions $\{c_t(z^t), l_t(z^t), k_t(z^t)\}_{t=0}^{\infty}$, prices $\{w, r\}$, aggregate capital and labour $\{K_t, L_t\}_{t=0}^{\infty}$, and a measure of agents $\{\mu_t(k, z)\}_{t=0}^{\infty}$ are a balanced-growth path of the original economy. Where the relationship between the growth-adjusted economy variables and those of the original economy are given by equations (28)-(35).*

Proof. The proof proceeds in four parts, corresponding to the four components of the equilibrium definitions: (i) the household optimizes, (ii) aggregates are determined by individual actions, (iii) markets clear, and (iv) a condition on the distribution of agents. In all four part the proof involves proving that an equilibrium solution for one definition also satisfies that condition for the other definition via the renormalization.

Part (i) - the household optimizes: since the return functions are strictly concave and the choice sets are strictly convex the combination of the first order conditions, the slackness conditions, and the transversality conditions together are both necessary and sufficient conditions for household optimization. In what follows I go one-by-one through these conditions from the households problem in the original economy and show that substitution based on the renormalization defining the growth-adjusted economy gives the optimality conditions for the households problem in the growth-adjusted economy; these substitions could equally be reversed. I begin with the first order conditions of the household problem from the original economy (imposing constant r and w conditions of the

BGP), these are given by,

$$
\begin{aligned}
\beta^t \Pr(z^t) u_c(c_t(z^t), l_t(z^t)) &= \beta^t Pr(z^t)\lambda_t(z^t)(-1) \\
\beta^t \Pr(z^t) u_l(c_t(z^t), l_t(z^t)) &= \beta^t Pr(z^t)\lambda_t(z^t) w z_t E_t \\
0 = \beta^t \Pr(z^t)\lambda_t(z^t)(-1) + \sum_{z_{t+1}} [\beta^{t+1} Pr(z^{t+1})\lambda_{t+1}(z^t, z_{t+1})(1+r)] &+ \beta^t \Pr(z^t)\kappa_t(z^t)(-1) \\
\underline{\mathrm{k}}(E_t) - k_{t+1}(z^t) &\leq 0 \\
c_t(z^t) + k_{t+1}(z^t) &= w z_t E_t(1-l_t) + (1+r)k_t(z^{t-1}) \\
\kappa_t(z^t) &\geq 0
\end{aligned}
$$

for all z^t, for all t; where $\beta^t Pr(z^t)\lambda_t(z^t)$ are the lagrange multipliers associated with the budget constraints, equation (22), and $\beta^t Pr(z^t)\kappa_t(z^t)$ are the lagrange multipliers associated with the borrowing constraints, equation (23). Define $\hat{\lambda}_t(z^t) = E_t^{\theta-(1-\theta)(1-\gamma)}\lambda_t(z^t)$ and $\hat{\kappa}_t(z^t) = (1+g)^{(1-\gamma)(1-\theta)} E_{t+1}^{\theta-(1-\theta)(1-\gamma)}\kappa_t(z^t)$. Then by substituting all variables for their hatted equivalents using (28)-(35) and the definitions of $\hat{\lambda}_t(z^t)$ and $\hat{\kappa}_t(z^t)$ just given, we get

$$
\begin{aligned}
\hat{\beta}^t \Pr(z^t) u_c(\hat{c}_t(z^t), \hat{l}_t(z^t)) &= \beta^t Pr(z^t)\hat{\lambda}_t(z^t)(-1) \\
\hat{\beta}^t \Pr(z^t) u_l(\hat{c}_t(z^t), \hat{l}_t(z^t)) &= \hat{\beta}^t Pr(z^t)\hat{\lambda}_t(z^t) w z_t \\
0 = \hat{\beta}^t \Pr(z^t)\hat{\lambda}_t(z^t)(-(1+g)) + \sum_{z_{t+1}} [\hat{\beta}^{t+1} Pr(z^{t+1})\hat{\lambda}_{t+1}(z^t, z_{t+1})(1+r)] &+ \hat{\beta}^t \Pr(z^t)\hat{\kappa}_t(z^t)(-1) \\
\hat{\underline{\mathrm{k}}} - \hat{k}_{t+1}(z^t) &\leq 0 \\
\hat{c}_t(z^t) + (1+g)\hat{k}_{t+1}(z^t) &= \hat{w} z_t(1-\hat{l}_t) + (1+\hat{r})\hat{k}_t(z^{t-1}) \\
\hat{\kappa}_t(z^t) &\geq 0
\end{aligned}
$$

for all z^t, for all t. Which are the first order conditions of the households problem in the growth-adjusted economy. Where $\hat{\beta}^t Pr(z^t)\hat{\lambda}_t(z^t)$ are the lagrange multipliers associated with the budget constraints, equation (37), and $\hat{\beta}^t Pr(z^t)\hat{\kappa}_t(z^t)$ are the lagrange multipliers associated with the borrowing constraints, equation (38).

The slackness conditions for the original economy are given by

$$
\kappa_t(z^t)[\underline{\mathrm{k}}(E_t) - k_{t+1}(z^t)] = 0
$$

for all z^t, for all t. By substitution of variables for their growth-adjusted (hatted) equivalents, and canceling some variables we know to be non-zero, we get

$$
\hat{\kappa}_t(z^t)[\hat{\underline{\mathrm{k}}} - \hat{k}_{t+1}(z^t)] = 0
$$

for all z^t, for all t. Which are the slackness conditions for the households problem in the growth-adjusted economy.

The transversality condition for the original economy is given by

$$
\lim_{T\to\infty} \beta^T u_c(c_T(z^T), l_T(z^T)) k_{T+1}(z^T) = 0
$$

By substitution of variables for their growth-adjusted (hatted) equivalents we get

$$\lim_{T\to\infty} \hat{\beta}^T u_c(\hat{c}_T(z^T), \hat{l}_T(z^T))\hat{k}_{T+1}(z^T) = 0$$

Which is the slackness condition for the households problem in the growth-adjusted economy. Thus the renormalized solution of the households problem in the original economy solves the households problem in the growth-adjusted economy; and vice-versa. (Note: $l = 0$ is ruled out by utility function, I assume here that $l = 1$ is also suboptimal and so concentrate solely on interior solutions for l. The result of this Proposition can be extended to allow for $l = 0$.)

Part (ii) - aggregates are determined by individual actions: The aggregation conditions for the original economy are given by $K_t = \int k_t(z^t)d\mu_t(k_t(z^t), z^t)$, and $L_t = \int z^t E_t(1-l_t)d\mu_t(k_t(z^t), z^t)$. Simply substituting the variables for their hatted equivalents, and taking advantage of the property imposed on the distribution function μ_t as part of the definition of a Balanced-Growth Path we get the aggregation conditions of a stationary equilibrium of the growth-adjusted economy, namely $\hat{K} = \int \hat{k}_t(z^t)d\hat{\mu}(\hat{k}_t(z^t), z^t)$, and $\hat{L} = \int z^t(1-\hat{l}_t)d\hat{\mu}(\hat{k}_t(z^t), z^t)$. And vice-versa.

Part (iii) - markets clear: The market clearance for a Balanced Growth Path in the original economy are given by $r - (\alpha K_t^{\alpha-1}L_t^{1-\alpha} - \delta) = 0$ and $w - (1-\alpha)K_t^{\alpha}L_t^{-\alpha} = 0$ for all t. Simply substituting the variables for their hatted equivalents we get the market clearance conditions of a stationary equilibrium of the growth-adjusted economy, namely $\hat{r} - (\alpha\hat{K}^{\alpha-1}\hat{L}^{1-\alpha} - \delta) = 0$ and $\hat{w} - (1-\alpha)\hat{K}^{\alpha}\hat{L}^{-\alpha} = 0$

Part (iv) - a condition on the distribution of agents: The condition on the distribution of agents for a Balanced Growth Path in the original economy is that the measure of agents is invariant on a renormalization of the state-space by a constant function of aggregate endowment, namely

$$\mu(k_t(z^t)/E_t, z^t) = \int\int_{z^t \geq z^{t-1}} (k_t(z^{t-1})/E_t, z^{t-1})\mu(k^{t-1}/E_{t-1}, z^{t-1})Q(z_{t-1}, dz_t)$$

Simply substituting the variables for their hatted equivalents we get an invariant measure that satisfies the condition on the distribution of agents for a stationary distribution in the growth-adjusted economy, namely

$$\hat{\mu}(\hat{k}_t(z^t), z^t) = \int\int_{z^t \geq z^{t-1}} (\hat{k}_t(z^{t-1}), z^{t-1})\hat{\mu}(\hat{k}^{t-1}, z^{t-1})Q(z_{t-1}, dz_t)$$

Q.E.D.

Since computation of the model stationary equilibrium of the growth-adjusted model is based on the models recursive forumulation, the definition of Stationary (Recursive) Competitive Equilibrium (Definition 6) is now given in recursive value function iteration notation,

Definition 7. *A Stationary (Recursive) Competitive Equilbrium is an agents value function $V(k, z)$; agents policy functions $l = g^l(k, z)$ and $k' = g^{k'}(k, z)$; an interest rate r and wage w; aggregate capital K and labour H; and a measure of agents $\mu(k, z)$; such that*

1. *Given prices r & w, the agents value function $V(k,z)$ and policy functions $l = g^l(k,z)$ and $k' = g^{k'}(k,z)$ solve the agents problem*

$$
\begin{aligned}
V(k,z) = &\max_{l,k'} \left\{ u(c,l) + \beta \int V(k',z')Q(z,dz') \right\} \\
&s.t. \quad c + k' = wz(1-l) + (1+r)k \\
&\qquad\quad c \geq 0,\ k' \geq \underline{k}
\end{aligned}
$$

2. *Aggregates are determined by individual actions: $K = \int k d\mu(k,z)$, and $L = \int z(1-l)d\mu(k,z)$*

3. *Markets clear (in terms of prices): $r - (\alpha K^{\alpha-1}L^{1-\alpha} - \delta) = 0$.*

4. *The measure of agents is invariant,*

$$
\mu(k,z) = \int \int \left[\int 1_{k=k'(\hat{k},z)} \mu(\hat{k},z)Q(z,dz') \right] d\hat{k}dz \tag{40}
$$

where z is the labour supply shock which takes values in $Z = \{z_1, ..., z_{n_z}\}$ and evolves according to the *stationary* Markov transition function $Q(z,z')$.[35] Note that the wage is residualy determined by r.[36] The market clearance condition is more commonly expressed as $r = \alpha K^{\alpha-1}L^{1-\alpha} - \delta$, that the interest equals the marginal product of capital (minus the depreciation rate).

E Models Displaying Piketty's Third Law-of-Capitalism

Piketty (2014) describes *r-g* as an important driver of inequality. We now describe the 'Pareto inequality' models that Piketty (2014, 2015) refers to as capturing this idea.

These models exploit the mathematical property that exponential growth over a variable that is itself exponentially distributed will lead to a Pareto distribution. One can therefore derive analytical results about the behaviour of Pareto distributed variables. The Pareto distribution has the nice property of being fat-tailed, which is essential to be able to model income and wealth distributions. A fairly direct link between the models and the data is provided by the use of the *Inverted Pareto coefficient* in empirical work on inequality.

To model wealth we will follow the approach outlined in Jones (2015). An individual's wealth (asset holdings) will grow exponentially in age, and age will be exponentially distributed in the population, leading to Pareto distribution of wealth. Due to the need to manage continuous distributions the model will use continuous time, in contrast to the $inequality - reduces - r$ model which will be in discrete time. The model will be analysed on it's balanced-growth path.

[35] It is not necessary that Z be finite valued, or even bounded, but this simplifies the Markov theory required.

[36] The wage, which is given by the derivative of the Cobb-Douglas production with respect to labour, can be rewritten as a function of the interest rate and the parameters of the production function.

In laying out the $r\text{−}increases\text{−}inequality$ model we will use different notation for the individuals assets/wealth/capital a, and the aggregate capital level K. This is done as we wish to be able to use k as the capital per capita. Our focus here will be on a partial equilbrium model, although mention will also be made of the implications of general equilibrium.

Let a denote an individual's wealth at time t, which accumulates over time according to

$$\dot{a}(t) = ra(t) - \tau a(t) - c(t) \tag{41}$$

where r is the interest rate, τ is a wealth tax, $c(t)$ is consumption at time t.

Assume consumption is a constant fraction α of wealth, so

$$\dot{a}(t) = (r - \tau - \alpha)a(t)$$

So the wealth of an individual of age x at date t is

$$a_t(x) = a_{t-x}(0)e^{(r-\tau-\alpha)x} \tag{42}$$

where $a_{t-x}(0)$ is the initial wealth of a newborn at date $t - x$, which will be given as inheriting the average wealth of those who died in that same period, as described in more detail below.

We now turn to age distribution where we will make assumptions about births and deaths which will generate a population which has an exponential distribution in age. Assume that the number of people born at time t is

$$B(t) = B(0)e^{nt}$$

Assume that death is a Poisson process with arrival rate d. Then the stationary distribution for this birth-death process is an exponential distribution in age,

$$Pr[age > x] = e^{-(n+d)x} \tag{43}$$

The next step is to combine our equation for wealth as a function of age, (42), together with the age distribution, (43). But first we need to define the initial wealth of newborns. We assume that they equally inherit the wealth of the people who die in this economy:

$$a_t(0) = \frac{dK(t)}{(n+d)N(t)} = \frac{d}{n+d}k(t)$$

where K is aggregate capital (wealth) and k is capital per capita (ie. the average amount of capital per person).

Assume the economy is on a balanced growth path, so capital per person grows with exogenous technology at a constant rate g, so $k(t) = k(0)e^{gt}$. Then it follows that the inheritance (initial assets) of person of age x at time t (ie. born at time $t - x$) are

$$a_{t-x}(0) = \frac{d}{n+d}k(t-x) = \frac{d}{n+d}k(t)e^{-gx}$$

substituting this into our equation wealth as a function of age, equation (42), gives

$$a_t(x) = \frac{d}{n+d} k(t) e^{(r-g-\tau-\alpha)x} \tag{44}$$

which is now an exponential growth process for individual assets in terms of age.

Combining the exponential growth process for individual assets in terms of age, equation (44), with the exponential distribution for age, we get a Pareto distribution for individual wealth, namely

$$\begin{aligned} Pr[Wealth > a] &= Pr[age > x(a)] \\ &= e^{-(n+d)x(a)} \\ &= \left(\frac{a}{\left(\frac{d}{n+d}\right) k(t)} \right)^{-\frac{n+d}{r-g-\tau-\alpha}} \end{aligned}$$

The Inverted Pareto coefficient (aka. Pareto inequality index) is just the inverse of the exponent in this equation, namely

$$\eta_{wealth} = \frac{r-g-\tau-\alpha}{n+d}$$

This is the key result that inequality is increasing in r-g!

It is possible, by using an AK-production function, to impose general equilibrium on the model. Doing so means that dynamic efficiency considerations require that r-g is a constant.[37] Since g is exogenous it follows that r will be a fixed constant determined by model fundamentals; so the true drivers of inequality would then be such fundamental constants as n, d, and g, with some role for individual behaviour α and taxation τ. Whether the tax returns are rebated lump-sum or simply destroyed also plays some role. For a more comprehensive analysis see Jones (2015).

Summing up, in these models inequality as measured by the Inverted Pareto coefficient is

$$\eta_{wealth} = \frac{r-g-\tau-\alpha}{n+d}$$

so an increase in r-g leads to increasing inequality. This occours because r-g determines the rate at which assets will accumulate over time. A higher r-g allows individuals with higher assets to have higher incomes, and since savings are a fixed fraction of income, or at least not declining, leads to ever higher assets relative to other individuals.

[37] Intuitively, this is a result of the intertemporal Euler equation in the Ramsey growth model (Solow-Swan growth model with endogenous savings choice); our assumption of consumption as a constant fraction of assets can be seen as arising from a log-utility function.