Income inequality dynamics with constraints

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Abstract

Governments want to prevent inequality while maintaining economic efficiency. We investigate how an economy can satisfy both these constraints. We use the relative factor share as a proxy for inequality and study a representative agent model to see how this share, along with capital, consumption and government debt, depends on taxation. Whether the model's evolutions can be constrained is understood as a problem of viability theory – a mathematical theory of constrained dynamic systems – and so we compute the viability *kernels* corresponding to our constraints. These kernels explain both how policy makers should act and why they act as they currently do. (**JEL Classification:** D31, D33, E25, N17, N37)

Keywords: Factor income, inequality, viability theory

I Introduction

In his paradigmatic 'Equality and Efficiency: The Big Tradeoff', Okun (1975) describes egalitarian policies as like leaking buckets: governments can transfer wealth to reduce inequality, but when they do so some wealth will be lost. While this tradeoff between inequality and economic efficiency has produced a rich macroeconomic literature, an explicit framework for understanding the policy problem the tradeoff creates has not yet been produced. Governments will have opinions about the acceptable levels of efficiency and inequality, and the ultimate policy problem is whether these opinions are consistent – whether there are any economic states that are compatible with constraints on both efficiency and inequality. That is this problem which this paper begins to solve.

We solve this problem with viability theory, the mathematical theory of constrained dynamic systems. Viability theory determines a set of initial conditions – the viability kernel – from which a dynamic system can be controlled within constraints. Viability theory has found many applications in economics (we mention a few in Section III) but this is its first application to income inequality. If the viability *kernel* is not empty there will be economic states which satisfy the imposed constraints, from which the economy can be controlled to satisfy the constraints indefinitely.[1] We will compute kernels corresponding to different inequality constraints and identify the economic states which tighter constraints render non-viable, demonstrating the tradeoff between different economic states and inequality.

The focus of our model is the "relative factor share" – the ratio of net capital income to net labour income. We first test the relationship between income inequality and the relative factor share using New Zealand data from the post-war period. Having established the relevance of the relative factor share we use it as a proxy for income inequality. The relative factor share is an economic aggregate so its evolution can be understood through a representative agent model. The representative agent

[1]We speak here about deterministic viability. Viability theory can allow for uncertainty modelled by *tychastic* disturbances - see Aubin, Bayen, and Saint-Pierre (2011).

model's tractability affords us our computational techniques.

Readers might be wary of analysing inequality with a representative agent model: if an economy is unequal, agents are heterogeneous and thus no agent is 'representative'. It is true that pathological heterogeneity could render any economic model intractable. However Turnovsky and García-Peñalosa (2008) show that the aggregate behaviour of models with only wealth heterogeneity will be identical to the aggregate behaviour of a corresponding representative agent model. Thus, models of the aggregate income going to labour and capital can be robust to heterogeneity in the agents generating that income.

While we constrain inequality through constraining the relative factor share, we constrain efficiency by constraining consumption, capital stocks and debt: inefficient economies discourage capital formation, restrict consumption and increase government debt. We do this in a framework that helps understand when Okun's bucket *matters* – when the tradeoff between inequality and efficiency makes an economic state unacceptable. That framework yields useful results, demonstrating that it deserves to be at the core of inequality research in the future.

In our economy, concerns about inequality needn't contradict concerns about inefficiency. The viability kernels our constraints produce show that many economic states remain viable when inequality is constrained, but that some – those with low capital stocks – do not. Moreover we can look at how the viability kernels depend on initial levels of debt – interestingly, we find no tradeoff between debt reduction and inequality reduction. In fact, debt reduction requires high tax rates which will naturally reduce inequality anyway. In the long run, the stabilising paths our kernels generate suggest that economies will have fairly similar levels of inequality, despite other economic variables stabilising at a wide range of steady states. These paths also demonstrate how policy makers can gradually reduce inequality while obeying other constraints.

In Section II we discuss how other researchers have conditioned inequality on economic variables, and we also show that the correlation between the relative factor share and inequality is strong, at least in New Zealand and particularly for the income inequality between a society's wealthiest and its masses. Section III explains viability theory and how it conceives policy problems. The model, its parametrisation and its interpretation in viability theory are explained in Section IV. Section V discusses the viability kernels we produce. Section VI compares the trends that our simulations follow as they are controlled within our constraints. Section VII asks how an economy can reduce its relative factor share while remaining within other constraints. Section VIII concludes. In an appendix we explain the numerical method which computed our kernels.

II Socio-economic background of our model

1 The context of capital in the 21st Century

Since Ricardo and Marx the rhetoric of inequality has focused on the struggle between capital and labour. To be wealthy is to be landed, to be a rentier. To be poor is to be landless, to be a worker. But of course there are both rich workers and poor workers, chief executives and manual labourers, and a pensioner living off their wealth might laugh at any suggestion of riches. If the distributions of labour income and capital income were identical – perhaps because people save only to smooth their consumption into retirement – an increased relative factor share would not increase income inequality. As it happens, capital tends to be distributed much less equally than wages (Piketty, 2014, tables 7.1, 7.2). As capital income and wages are positively correlated (the top 1% of capital earners are more-or-less the same people as the top 1% of labour earners) this results in a simple positive relationship between the relative factor share and inequality (Piketty, 2014, pp. 254 - 255).[2] We test the relevance of the capital-labour struggle to the history of income inequality in

[2] When capital income and wages are negatively correlated – as they were in the 19th Century and as they are in contemporary models like Turnovsky and García-Peñalosa (2008) – an increase in the relative factor share increases the income of some high-earners but not others, leading to a relationship less easily summarised.

New Zealand in Subsection 3.

In focusing on the shares of income taken by capitalists and labourers our work echoes that of Piketty (2014). Piketty shows that the ratios of capital to income have shifted significantly since the 19th century. In 1870 Britain, capital was worth almost 700% of national income (Piketty, 2014, Technical Appendix Table SI.2). By the end of World War Two it was only worth 313%, and by 2010 it had rebounded to 522%. These changes are representative of changes across the West because the shocks that caused them – ideological, military, economic – were as well. From our perspective, their importance comes from their relationship to shifts in the proportion of income going to capital. While an increasing capital stock will reduce the marginal product of capital and so reduce interest rates, Piketty believes that this effect will not be sufficiently strong to keep capitalists' share of income constant (Piketty, 2014, p. 220). In more technical terms, Piketty believes the elasticity of substitution between capital and labour is greater than 1.

In our research we will consider how policy can constrain inequality. We will focus on the functional side, on the ratio of returns to labour with returns to capital. Further, we assume an elasticity of substitution equal to 1. Piketty would contest this assumption so it deserves some justification. First, elasticities of substitution appear to be lower over the short term than the longer term Tipper (2012). Thus our paper can be thought of as considering Piketty's questions over a shorter timeframe. Second, it is worth noting that empirical analyses of the elasticity of substitution have consistently found elasticities less than one, making an elasticity = 1 a fairly conservative assumption.[3] It is the elasticity of substitution = 1 that implies the Cobb-Douglas production model we use below.

[3] For example, see Szeto (2001), Hall and Scobie (2005) and Tipper (2012) for New Zealand examples.

2 Others' modelling

Piketty was hardly the first economist to study inequality. The inequality literature is substantial, here we discuss a few of its more prominent papers to contextualise our own research.

One recurring topic is whether income inequality helps or hinders economic growth. While Okun (1975) theorised a tradeoff between inequality and efficiency, strictly speaking this needn't imply a tradeoff between inequality and *growth*, and indeed the empirical literature is inconclusive. Frank (2009) examines a panel of American states and finds that growth is associated with inequality, while Cingano (2014) examines a panel of OECD countries and finds that inequality decreases growth. Cingano (2014) suggests inequality stops some people from receiving an education, constraining the human capital formation economic growth requires. An alternative explanation is offered by Ranciere and Kumhof (2010), who show that financial crises tend to follow increases in income inequality and increases in poorer households' financial leverage. That paper constructs a model in which these features arise endogenously as a result of a shift in bargaining powers over incomes.

McManus (2013) places agents with differing access to capital markets inside a simple DSGE model. Following a recessionary shock, those with full access to capital markets continue gaining welfare but not those whose credit is constrained. This suggests financial reforms that aim at stabilisation of capital markets might decrease income inequality. Christopoulos and McAdam (2014) use gross and net Gini coefficient to capture inequality and a financial reform index. When inequality is measured gross of taxes and transfers, financial reforms seem not to have stabilised inequality; for net inequality there is evidence that they might. The authors contend that financial reforms complement a redistributive fiscal system.

Our model has no credit constraints built into it, but by constraining capital stocks in a closed economy we effectively constrain household borrowing. Thus our model tests the effect of household borrowing constraints indirectly.

Perhaps most relevant for our research are García-Peñalosa and Turnovsky (2007); Turnovsky and

García-Peñalosa (2008); García-Peñalosa and Turnovsky (2011); Turnovsky and Mitra (2013) and Karacaoglu (2015). Those papers study heterogeneous agents within a dynamic model, analysing how that heterogeneity evolves over time. By explicitly defining the nature of the heterogeneity – whether caused by differing preferences, wealth or skills – those papers can closely study the mechanisms which contract or expand overall inequality. This analytical depth becomes a weakness when a model cannot be solved explicitly and must be simulated on a computer; when agents' decisions about saving and labour supply differ, each agent's decision must be simulated separately. Heterogeneous agent models can take a prohibitively long time to compute and, in particular, a heterogeneous agent model would render the computational techniques we use in our viability analysis impractical.

In fact it was Karacaoglu (2015) who motivated us to model constrained inequality with viability theory. Karacaoglu (2015) asks how a social planner can constrain inequality and other social variables while the economy follows a feasible trajectory. This is the type of question viability theory answers.

As will be evident, our model describes an economy over the medium term: it explains changing levels of capital and debt, but not changing technology. Our total factor productivity is constant and so our results cannot be compared to those from longer term studies, such as the Kuznets curve of Kuznets (1955).

3 Top incomes and the relative factor share

In this subsection we study the empirical relationship between inequality and the relative factor share, to justify our use of the relative factor share as a proxy for inequality.[4]

Our measure of income inequality is the share of income taken by the top 10%, the top 1% and the top 0.1% of earners. The study of income inequality is the study of a distribution, which no single

[4]See Krawczyk and Townsend (2015b) for some details.

statistic can fully describe. As demonstrated by Atkinson (1970) this is not a trivial concern. In his comparison of twelve countries, India is the third most equal by one measure and the eleventh by another.

Income shares are justified first by their easy interpretation. The Pareto coefficient (which parametrises the distribution to a Pareto distribution) and the ever-present Gini coefficient (which compares the distribution's Lorenz curve to one of complete equality) lack that easy interpretation, and shifts in them can correspond to shifts anywhere along the distribution. Percentile ratios – say, the cut-off to be in the top quintile divided by the cut-off to be in the bottom quintile – are popular with government statistical agencies, probably because they can be easily calculated from survey data. However they ignore the tails of the distribution and so ignore the extremes which correspond to the most visible inequality. It is the incomes of the very wealthy compared to the rest which has inflamed popular rhetoric.[5]

The more expedient reason why we use income share data is that it is available. We use data from the the World Top Incomes Database (Alvaredo, Atkinson, Piketty, and Saez, 2014), which in turn is from Atkinson and Leigh (2007). They calculated the share of total income flowing to top percentiles by comparing tax data to national accounts data. Their data series is from 1921 to 2011 (though it is punctured by missing years and the 0.1% share series ends in 1989). As their estimates of high incomes are from income tax data, they are affected by changes to the tax system. Quoting a 1953 New Zealand Census and Statistics Department report they tell us that "income-tax law is dynamic rather than static and there are few years in which amendments, some major and others minor, to the law have not affected the statistics". Thus year-to-year comparisons are unreliable and a regression in differences would be misspecified. The tax-treatment of transfer payments has changed over time, with all benefits being taxed from 1986. Atkinson and Leigh include transfers in total income in all years. However income is measured gross of tax.

We derive the factor ratio from the income measure of national accounts, for which we have data

[5] Remember the Occupy Movement's slogan? *We are the 99%.*

from 1939. Data from 1972 were obtained with Statistics New Zealand's Infoshare tool, older data was obtained from the latest revision available in the New Zealand Official Yearbooks.

Table 1: Summary statistics and unit root tests

	relative factor share	10% share	1% share	0.1% share
mean	0.684	30.812	7.601	1.575
standard deviation	0.130	2.373	1.428	0.325
number of years	73	68	68	45
final year	2011	2011	2011	1989
ADF test statistic	-2.235 **	-2.901 ***	-1.772 **	-1.463 *
ADF lags	1	1	2	1

P-values follow $*** < 0.01 < ** < 0.05 < * < 0.1$. The number of lags in the ADF test were selected with the Akaike information criterion; the ADF tests include a drift term but do not include a time trend.

Table 1 summarises our data and tests for unit roots. For all our series the Augmented Dickey-Fuller (ADF) test statistics are significantly less than zero, suggesting that the correlation between subsequent observations is significantly less than one.[6] This justifies our regression analysis. However the evidence that the 0.1% series is stationary in particular is somewhat weaker than that for the other series; its ADF null hypothesis is only rejected with 90% confidence. This is what we might expect given the very wealthiest receive much of their wealth from financial assets. Thus for the sake of robustness we complement our regression analysis with a cointegration test.

Table 2 includes the results of both our regression analysis and a cointegration test, with plots of income shares against the relative factor share in Figure 1. All three of our estimates are positive, significant and able to explain a substantial proportion of the variation in inequality. This is strong

[6]The ADF test is the standard unit root test. It helps to discover whether the autoregressive process generating our data has a unit root. See Enders (2008) for a fuller treatment.

Table 2: The relationships between inequality and the relative factor share

	10% share	1% share	0.1% share
effect of relative factor share	6.534 ***	6.660 ***	1.624 ***
	(2.341)	(1.307)	(0.302)
R^2	0.136	0.389	0.578
number of years	68	68	62
ADF test on residuals	-3.406 ***	-3.709 ***	-3.481 ***
ADF lags	1	1	1

Robust standard errors in brackets. P-values follow $*** < 0.01 < ** < 0.05 < * < 0.1$. The number of lags in the ADF test were selected with the Akaike information criterion; the ADF tests have neither drift terms nor time trends.

evidence that *some* relationship exists between the relative factor share and inequality. Further, as we move up the income distribution our estimates are more significant and able to explain an increasing proportion of variation. This is what we would hope: while inequality towards the middle of the distribution may still be caused somewhat by wage inequality, at the extreme end much income is from capital and so increases in their wealth will be reliant on greater returns to capital. In fact the parameter estimate on the 10% share is similar to that on the 1% share. As both have as their units 'percentage points of total income', this suggests that when the relative factor share increases the income of the top 10%, it is only really increasing the income of the top 1%. The same cannot be said for the top 0.1%, but their estimate is more than a tenth of the 1% parameter and thus they do also benefit disproportionately.

If our series did follow a unit root, the augmented Dickey-Fuller tests in Table 2 would provide an alternate justification for causality between inequality and the relative factor share. The null hypotheses that the residuals follow a unit root is rejected for all three regressions, suggesting that the residuals are $I(0)$. If the inequality and relative factor share series are indeed $I(1)$, they must be cointegrated.

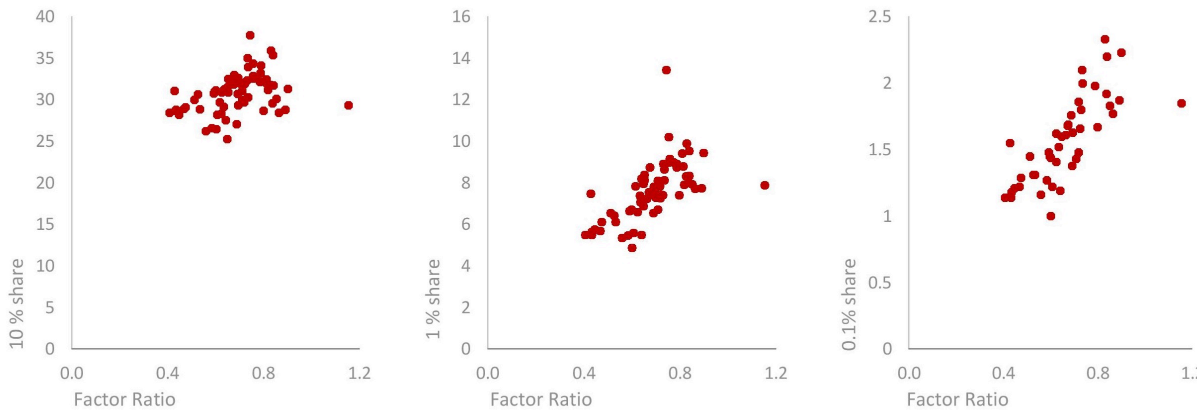

Figure 1: 10%, 1% and 0.1% income shares, vs. the relative factor share.

III Viability theory

Viability theory is the mathematics which studies constrained dynamic systems. A system's evolution is *viable* if the system remains, for the entire time of the evolution, within a constraint set K. The *viability kernel* is a subset of K which contains all points which can be made viable, given a constrained control set U. Viability theory attempts to determine whether a nonempty viability kernel exists and, if so, what its boundaries are.

Viability theory formalises the 'satisficing' policies of Simon (1955) – so long as viability is not threatened, any policy is good enough. This characterisation provides a good description of real world decision-making. For example, an inflation-targeting central banker will avoid changing interest rates until doing so is necessary. This will be more naturally expressed as a viability problem than as an optimisation problem, and management theories based on viability will more accurately describe how managers actually behave.

Most viability theory applications have focused on environmental policy – see for example Béné, Doyen, and Gabay (2001), Martinet and Doyen (2007), De Lara, Doyen, Guilbaud, and Rochet (2006), and Martinet, Thébaud, and Doyen (2007). Viability theory has also been applied to finance (see Pujal and Saint-Pierre (2006)), managerial economics (see Krawczyk, Sissons, and Vincent (2012)), macroeconomics (see Clément-Pitiot and Saint-Pierre (2006), Clément-Pitiot and Doyen (1999), Krawczyk and Kim (2009), Krawczyk and Kim (2014), Bonneuil and Saint-Pierre (2008), Bonneuil and Boucekkine (2008), Krawczyk and Kim (2004), Krawczyk and Sethi (2007)) and microeconomics (see Krawczyk and Serea (2013)).

Rigorous introductions to viability theory can be found in Aubin (1991), Quincampoix and Veliov (1998), Veliov (1993) and Aubin et al (2011). Here we present only these notions of viability theory which are essential to our analysis.

In viability theory, the differential inclusion

$$\dot{x}(t) \in F(x(t)) \tag{1}$$

is the basic description of a dynamic system. It states that at $x(t)$ the change in the system's state – its velocity – will be a member of $F(x(t))$, where F is a set-valued map from system states to sets of possible velocities. In control theory the map F has the form $F(x) = f(x, U) = \{f(x, u); u \in U\}$, where $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is a continuous vector-valued function representing the system's equations of motion and U is a compact set in $\mathbb{R}^m$.[7] In this case, we can re-write (1) as

$$\dot{x}(t) = f(x(t), u(t)) \tag{2}$$

$$u(t) \in U(x(t)) \tag{3}$$

where (2) is a standard parameterised differential (vector) equation and (3) states that the control choice $u(\cdot)$ must come from a set $U(x(\cdot))$, which may be state-dependent.

As above, let K represent the closed set of constraints that state $x(t)$ must satisfy for all t – say,

[7] For other interpretations of (1) see Krawczyk and Pharo (2013).

an inflation and output-gap constraint. Given a set-valued map $F : K \rightsquigarrow \mathbb{R}^n$, we say that $x_0 \in K$ is *viable in K under F* if there exists at least one solution to the following system:

$$\forall t \in \Theta \begin{cases} x(t) \in K, \\ \dot{x}(t) \in F(x(t)), \end{cases} \tag{4}$$

that starts at $x(0) = x_0$ and remains in K forever: $\Theta \equiv [0, \infty)$.[8]

Formulation (4) describes the viability of an individual system state. The viability kernel $\mathcal{V}_F(K)$ is the set of all viable states:

$$\mathcal{V}_F(K) \equiv \{x(0) : \exists\, x(t)\, \text{satisfying (2)-(3) and constraints } K\, \forall t\}. \tag{5}$$

For a control problem, the viability kernel $\mathcal{V}_F(K)$ is the area in which a control exists which can keep the system within K indefinitely. If a trajectory begins inside the viability kernel $\mathcal{V}_F(K)$ then we have sufficient controls to keep this trajectory in the constraint set K for all t. If a trajectory begins outside the kernel then it will inevitably leave K. The viability kernel $\mathcal{V}_F(K)$ has important implications for policy: it allows us to construct control rules that maintain the system's viability.

To illustrate viability we reproduce Figure 2 from Krawczyk and Pharo (2013). The state constraint set K is represented by the yellow (lighter) shape contained in state space X. The solid and dashed lines symbolise system evolutions, which converge to where the arrows end. The brown (darker) shape is the viability kernel. The trajectories that start in the kernel remain in K and are thus viable. The trajectories that start outside the kernel will eventually leave K.

We will say that the system's dynamics $F(\cdot)$ are *compatible* with K if there exist a set of economic states from which there exist viable evolutions that can obey the constraints indefinitely. The largest such set is the viability kernel $\mathcal{V}_F(K)$.

[8] Viability is normally defined in terms of an infinite time horizon, but this is not necessary.

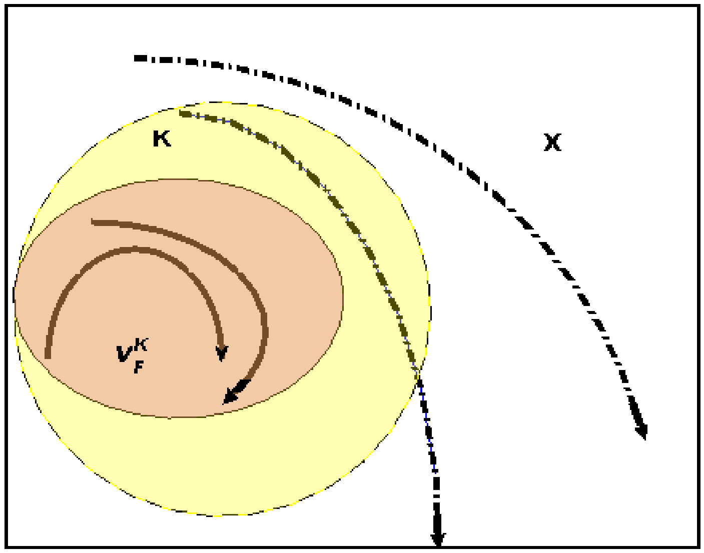

Figure 2: The viable and non viable trajectories for a time-invariant dynamic system.

IV Formulating the social planner's viability problem

1 The socio-economic modelling

In this sub-section we derive the vector-differential inclusion $F(\cdot)$ through which our economy evolves. We then derive the closed-form expression for the relative factor share.[9]

Let output be y, capital k and labour l. Following Judd (1985, 1987) and Krawczyk and Judd (2015), our model includes a representative producer with a Cobb-Douglas production function ($y = Ak^{\alpha}\ell^{1-\alpha}$), an infinitely lived representative household who enjoys consumption $\left(u(c) = \frac{c^{1-\gamma}}{1-\gamma}\right)$ and prefers not to work $\left(v(\ell) = V\frac{\ell^{1+\eta}}{1+\eta}\right)$, and a government which taxes income (τ) to finance its debt (B) and exogenous spending (G).[10] Capital depreciates at rate δ. Our household discounts

[9] This section is based on earlier work in Krawczyk and Townsend (2015a).

[10] Judd (1987) allows labour and capital income to be taxed at different rates. Testing the implications of this would make for an interesting extension to our analysis.

future consumption at rate ρ. Labour supply and savings are both endogenous, allowing each economic state to be characterised by the state vector $x(t) = (k(t), c(t), B(t), \tau(t))$ where t is time as in Section III.

We propose that the social planner endeavours to establish whether the system dynamics $F(x(t))$ are compatible with the viability constraints K:

$$K \equiv \left\{ (k, c, B, \tau) : \left| \begin{array}{c} \underline{k} \le k(t) \le \overline{k} \\ \underline{c} \le c(t) \le \overline{c} \\ \underline{B} \le B(t) \le \overline{B} \\ \underline{\tau} \le \tau(t) \le \overline{\tau} \\ 0 \le \chi \le \overline{\chi} \end{array} \right. \right\} \tag{6}$$

where the constraints on k, c, B, τ, χ – $\underline{k}, \overline{k}, \underline{c}$, *etc.* – will be explained in the calibration Section 2.

If the system's dynamics are compatible with K, there will exist a set of economic states from which there exist viable evolutions that can obey the constraints indefinitely. So, in the parlance of Section III, the social planner wants to compute the viability kernel $\mathcal{V}_F(K)$.

Capital accumulation is equal to output less depreciation, consumption and government expenditure:

$$\frac{dk}{dt} = Ak^{\alpha}\ell^{1-\alpha} - \delta k - c - G\,. \tag{7}$$

If $\lambda > 0$ is the marginal value of capital at time t, then it follows from maximization of the utility function $u(c) - v(\ell)$ on an infinite horizon that[11]

$$\frac{d\lambda}{dt} = \lambda(\rho - \bar{r})\,. \tag{8}$$

[11]The private marginal value of capital λ is the adjoint state in the perfect-foresight household utility $u(c) - v(\ell)$ maximisation problem. See Judd (1987); Brock and Turnovsky (1981) for details.

Here, $\bar{r} = (1-\tau)\left(\frac{\partial y}{\partial k} - \delta\right)$ is the after tax marginal product of capital. Expanding $\bar{r}$ in (8) yields

$$\frac{d\lambda}{dt} = \lambda\left(\rho - (1-\tau)\left(\alpha A\left(\frac{\ell}{k}\right)^{1-\alpha} - \delta\right)\right). \tag{9}$$

Government debt will grow proportionate to those interest rates, while also growing in $G - T$:

$$\frac{dB}{dt} = \bar{r}B + G - T \tag{10}$$

where T is tax. If we allow capital depreciation to be tax-deductible the expression for tax becomes

$$T \;=\; \tau k\frac{\partial}{\partial k}\Big(y - \delta k\Big) + \tau\ell\frac{\partial y}{\partial \ell} = \tau A k^{\alpha}\ell^{1-\alpha} - \tau\delta k \tag{11}$$

Combining the last two expressions results in the debt dynamics

$$\frac{dB}{dt} = \bar{r}B - \tau A k^{\alpha}\ell^{1-\alpha} + G + \tau\delta k\,. \tag{12}$$

That is, debt can diminish if output is large or if the tax rates are high.

The marginal utility of consumption can be expressed as both $\frac{du}{dc} = \frac{1}{c^{\gamma}}$ and as $\lambda = \frac{du}{dc}$. Hence,

$$c = \frac{1}{\lambda^{1/\gamma}}\,, \tag{13}$$

which after differentiation[12] yields

$$\frac{dc}{dt} = \frac{-1}{\gamma}\cdot\frac{1}{\lambda^{1+1/\gamma}}\cdot\frac{d\lambda}{dt} = \frac{-1}{\gamma}\,c^{1+\gamma}\,\frac{d\lambda}{dt}\,. \tag{14}$$

Using (9), after some simplifications, we get

$$\frac{dc}{dt} = -\,c\cdot\frac{\rho + \left(\delta - \alpha\,A\,k^{\alpha-1}\,\ell^{1-\alpha}\right)\,(1-\tau)}{\gamma} \tag{15}$$

Consumption has a trivial steady state and will grow to reach a positive steady state if the discount rate or depreciation are small or output is large.

To fully describe our states' evolutions, the differential equations for capital, consumption and government debt (7), (15), (12) need to be completed by a differential *inclusion* for the tax rate τ:

$$\frac{d\tau}{dt} = u \in [-d,\, d],\; d \geq 0 \tag{16}$$

[12] The adjoint state $\lambda(t)$ will not 'jump' because $x(t)$ will not be on the boundary of K.

This requires the government adjusts tax rates smoothly.

Labour in (7), (15), (12) is endogenous and can be expressed as a function of capital and consumption: Wages w equal the marginal product of labour:

$$w = \frac{dy}{d\ell} = \frac{(1-\alpha)k^{\alpha}\, A}{\ell^{\alpha}} \tag{17}$$

In equilibrium, the marginal rate of substitution between leisure and consumption should be equal to the real wage:

$$\frac{(1-\tau)w}{c^{\gamma}} = \ell^{\eta} V\,. \tag{18}$$

Substituting wages and solving for labour yields

$$\ell = \left(\frac{(1-\tau)(1-\alpha)Ak^{\alpha}}{c^{\gamma}V}\right)^{\frac{1}{\alpha+\eta}}\,, \tag{19}$$

which indeed depends only on capital and consumption.

We can now use (19) to substitute labour into (7), (15), (12). The resulting expressions define the evolution of our economy in terms of k, c, B, τ. Those expressions are cumbersome and elucidate no real intuition, so we won't state them explicitly. Together with (16), they describe the economy for which Krawczyk and Judd (2015) produced viability kernels, showing moderate tax adjustments guaranteed a balanced economy. We extend that work by computing and constraining the relative factor share as a proxy for constrained inequality.

The relative factor share is the ratio of net capital income to net labour income, $\chi \equiv \bar{r}k/\bar{w}l$ where $\bar{r}$ and ℓ have already been derived; $\bar{w}$ is wages multiplied by $1-\tau$. Expressing χ in terms of k and c, we have:

$$\chi \equiv \frac{k\bar{r}}{l\bar{w}} = \frac{\alpha}{1-\alpha} - \delta\left(\left(\frac{Vc^{\gamma}}{1-\tau}\right)^{1-\alpha} k^{\eta(1-\alpha)}(A(1-\alpha))^{-(\eta+1)}\right)^{\frac{1}{\alpha+\eta}} \tag{20}$$

As we explained in Section 3, this ratio correlates with income inequality. When $\delta = 0$ it simplifies to a constant, suggesting that in a one-tax model without depreciation χ would depend only on α, the Cobb-Douglas output elasticity. Given α, when $\delta > 0$, the relative factor share will depend on both the direct effects of taxation and the effects of taxation on capital and consumption. If higher taxes decrease capital and consumption, the long run effects of higher taxes on the relative factor share could be ambiguous: if we want to study them, a computational simulation is required.

2 Calibration

Following Krawczyk and Judd (2015), this paper analyses kernels produced for a "reasonably industrialized economy composed of rational agents interested in the near future, drawing a fair satisfaction from consumption and feeling, quite strongly, the burden of labor". We assume $\rho = 0.04$, $\alpha = 0.43$, $\eta = 1$ and $\gamma = 0.5$. In contrast to Krawczyk and Judd (2015) where $\delta = 0$, we assume $\delta = 0.05$. When $\delta = 0$ tax has no impact on χ, see (20).

Using a stylised steady state $k^s = \ell^s = 1$ with no taxes and no government expenditure, we calibrate A and V and obtain $A = 0.2093$, $V = 0.2989$. We then assume that government expenditure G is constant and set at 10% of no-tax steady-state output; $G = 0.1 \cdot A = 0.0209$.

The constraints come from a combination of positive and normative sources, as well as from the requirement to close K. For example, the lower bound on capital might be tied to a normative requirement concerning the nation's GDP, whereas the upper bound might be based only on the observation that capital would never realistically fluctuate that far from its steady state.

In our model, capital will be within 10% and 200% of no-tax steady state capital stock, $k \in [0.1, 2]$. Consumption should not deviate too far from a long-run equilibrium $c \in [0.0150, 0.225]$, compare Krawczyk and Judd (2015). Debt may grow to 350% of the maximum steady-state capital stock and also drop somewhat below zero, $B \in [-1, 3.5]$. The tax rate cannot be less than zero, and can at most be equal to 80%, $\tau \in [0. 0.8]$. The tax-rate adjustment speed – the amount the regulator increases or decreases the tax rate within a year – will be less than 20 percentage points, $u \in [-0.2, 0.2]$.

We will first require χ to be ≤ 0.4, and then next require it to be ≤ 0.25. The relative factor share $\chi = 0.4$ corresponds per our OLS analysis in Section II to the top 1% taking 5.7% of income and to the top 0.1% taking 1.1% of income. The value of $\chi = 0.25$ corresponds to the top 1% taking 4.7% and to the top 0.1% taking 0.9%. (In New Zealand the top 1% currently take about 8% of national income. In the mid-1980s they were taking about 5.5%.)

We also require χ to be positive. This is less a normative constraint, more an interpretation aid. Viability theory finds points from which a system can be kept within certain bounds. (This does not require that the system be kept at those points.) Negative χ requires negative interest rates, as wages (the marginal product of labour) will be positive in a Cobb-Douglas production function. Negative interest rates require $\dfrac{\partial Y}{\partial k} < \delta$. Our simulations confirm that such a situation is not sustainable: capitalists will not invest if they are receiving a negative return. Thus requiring $\chi > 0$ removes points which are viable but not the result of any long-run steady state, simplifying our analysis.

Thus the constraint set K for one tax, for which we will find the viability kernel, is

$$K = [0.1,\, 2] \times [0.0150,\, 0.225] \times [-1,\, 3.5] \times [0,\, 0.8] \times [0,\, \overline{\chi}]\,, \tag{21}$$

where $\overline{\chi}$ is either 0.4 or 0.25.

V Viability kernel comparison

In this section we analyse the viability of relative factor share constraints. We first ask how a relative factor share constraint shrinks the viability kernel. If there is a sharp tradeoff between equality and efficiency, the relative factor share constraint will render many economic states non-viable, shrinking the viability kernel. If there is no significant tradeoff between equality and efficiency, the relative factor share constraint will not significantly affect the viability kernel.[13] We have computed the kernels with VIKAASA, see Appendix VIII. Second, we will ask how government debt affects our conclusions. (These questions are a sample from the many viability theory could answer –

[13]This section is based on earlier work in Krawczyk and Townsend (2015c).

we could have just as easily asked whether requiring smoother tax adjustments shrinks the kernel significantly, whether higher consumption than the current $\bar{c}$ is viable, whether economies with high elasticities of substitution shrink more quickly when χ constraints are imposed, and whether the ability to tax capital and labour income at different rates expands the kernel.) [I've tried to make this emphasise the power of viability theory while not leaving us sensitive to the complaint 'well, why didn't you answer those questions?' By listing many questions I achieve both those goals.]

Figure 3 shows 3D kernel slices for the kernels produced by three different sets of constraints.[14] As discussed above, all three slices require $\chi \geq 0$. The first has no further constraint. The second requires $\chi \leq 0.4$. The third requires $\chi \leq 0.25$. These slices' projections onto the consumption-capital plane are shown in black.

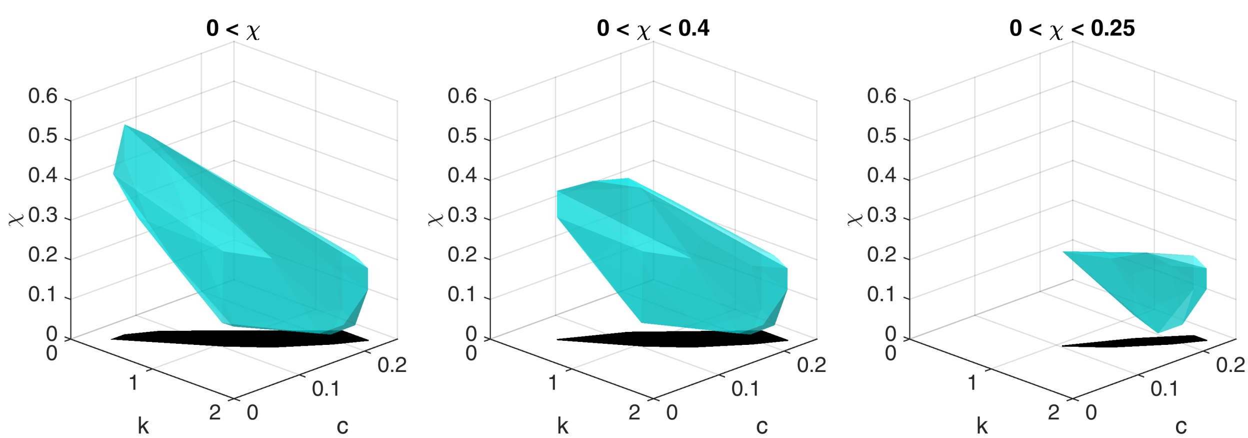

Figure 3: Viability kernels for different relative factor share constraints

The constraint shrinks both the kernels and their projections. The lowest levels of capital become non-viable when the relative factor share is required to be less than 0.4. There is a much larger reduction in the kernel when the constraint is lowered to $\chi < 0.25$. At this point, only high levels of capital are viable. This suggests that economies with little capital may find themselves unable

[14] The figures are 3D slices of the 5D kernels. Those in Figure 3 include points regardless of their initial debt level, provided that government debt could be controlled to remain in $[-1, 3.5]$, whereas those in Figures 4 and 6 require debt to start at some level.

to constrain inequality, and that in general the tradeoff between equality and efficiency can be significant.

We now ask whether low inequality targets remain viable when a government has significant debt. Figure 4 requires debt to start equal to 3.5.

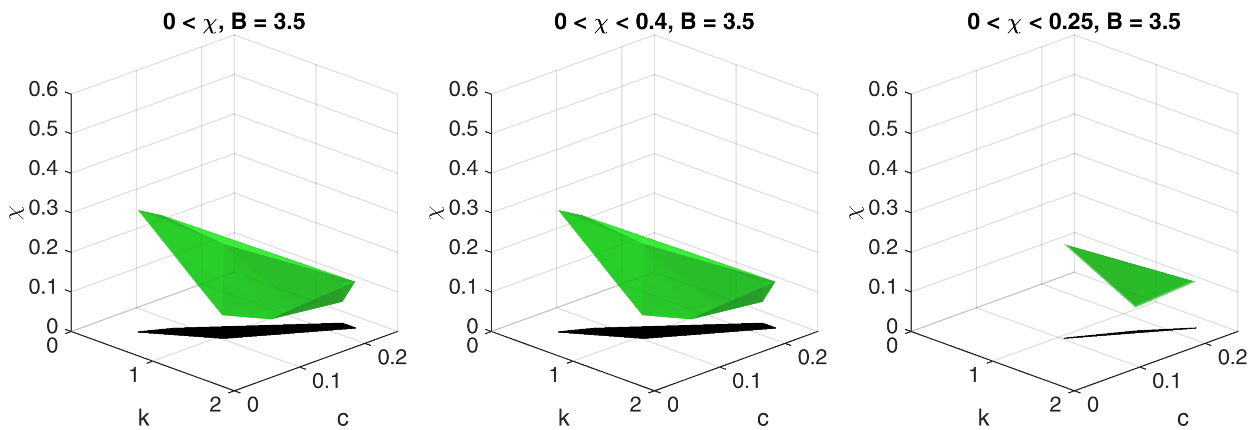

Figure 4: Viability kernels with high debt

The high-debt kernels are noticeably different to those in Figure 3. In particular, a high relative factor share is impossible with high levels of government debt. The left-most panel has no constraint on the relative factor share, but nonetheless the factor share is always < 0.4.

To see why high debt prevents a high relative factor share (and so prevents high inequality) note that the relative factor share isn't directly a function of debt. However, the relative factor share is decreasing in tax rates, suggesting high tax rates reduce capital income more than they reduce labour income (see (20)). Thus if high debt requires high tax rates then the corresponding relative factor share will be low. To demonstrate this, consider the converse case of low tax rates. Low tax rates will lead to a high χ, but are only viable with low initial debt. Set tax $= 0.2$, capital $= 0.2$ and consumption $= 0.02$. This corresponds to $\chi = 0.3549$. We impose only a bottom constraint $\chi \geq 0$ and compare the trajectories which begin with debt $= 0$ to those with debt $= 3$. We use VIKAASA to see which taxation strategy can keep the economy within the constraints on B, c, k and τ.

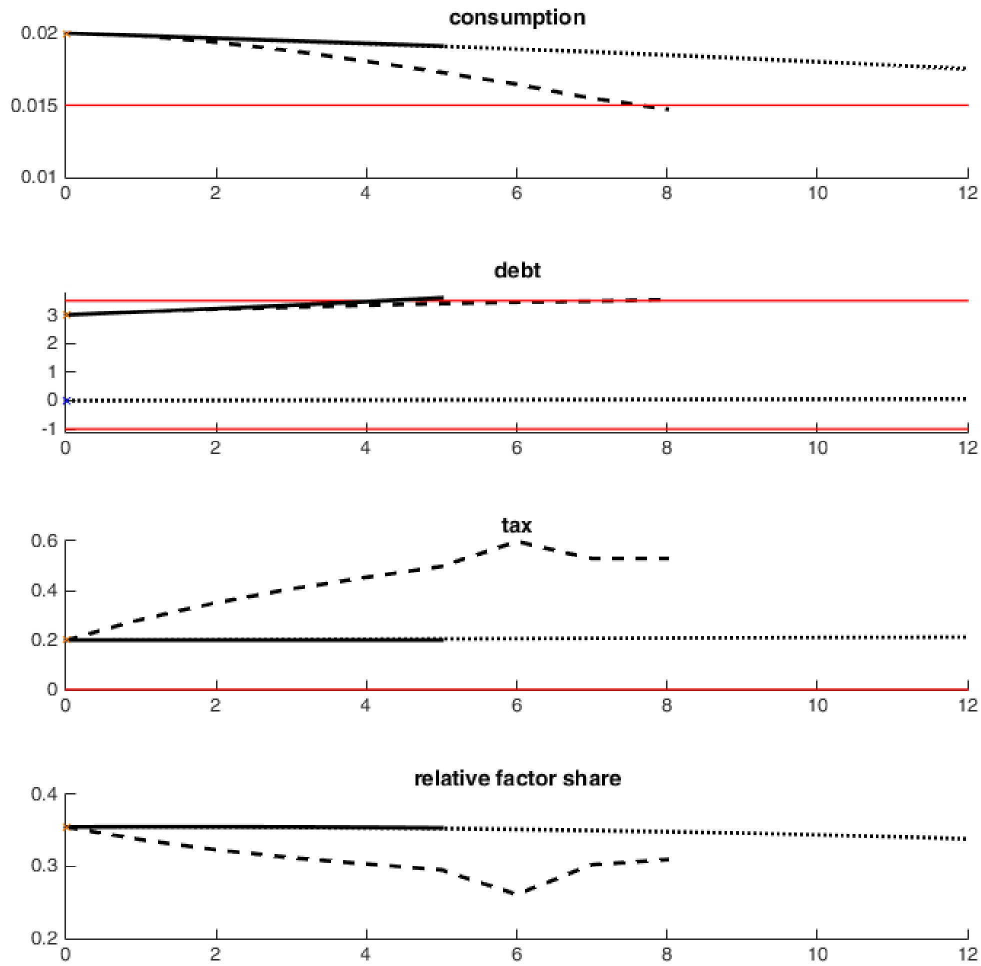

Figure 5: Time profiles of high debt (solid lines without tax changes, dashed with tax changes) and low debt (dots)

Figure 5 shows the time profiles of consumption, debt, tax and the relative factor share. If debt is high (dashed lines) we must increase taxes, otherwise the debt will crash through the upper bound. But increasing taxes (at full speed, to keep debt < 3.5) pushes consumption below its minimum constraint, showing us that the initial point is not viable. When we do not increase taxes (solid line) we go above the debt constraint. When we start from zero debt (dotted lines), debt can be stabilised without increasing taxes, allowing us to simultaneously stabilise consumption. We can see that high inequality is viable with low debt, but not with high debt.

Our viability kernels suggest that highly indebted economies will have neither high inequality nor low tax rates.

This does indeed seem to be the case. Japan has the highest public debt in the world, with public debt in 2010 equalling 206% of GDP (The World Bank, 2015). Our model predicts that Japan will have neither high inequality nor low tax rates: this is correct, Japan's 1% share in 2010 was 9.51% (Alvaredo et al, 2014) and its top marginal tax rate was 40% (National Tax Agency, 2010). In contrast the country in the Alvaredo et al (2014) database with the highest 1% share in 2010 was Columbia, where the wealthiest 1% take 20.45% of national income. Columbia had debt equal to only 38% of GDP (The World Bank, 2015) (the top tax rate was 33%).

The above analysis considered high-debt economies. In contrast, Figure 6 includes only viable points which start with debt equal to 0. The slice which required $\chi < 0.25$ had too few viable points to generate a 3D figure, and is thus excluded.

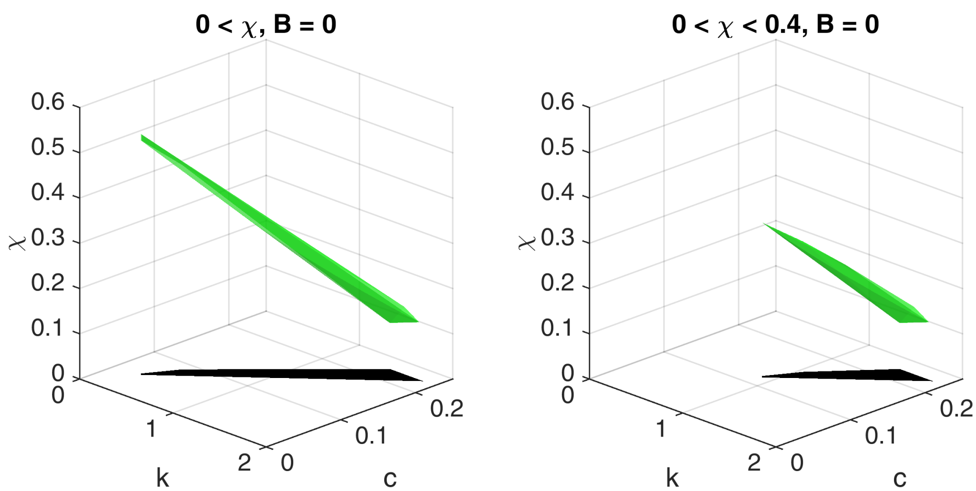

Figure 6: Viability kernels with debt = 0

The low debt kernels, both unconstrained and with $\chi \leq 0.4$, have far fewer viable points than their high debt counterparts in Figure 4. This is surprising: intuitively, lower debt would give governments more flexibility. The low-debt kernels are so small because of the bottom constraint on debt, $B \geq -1$. This constraint can be justified – perhaps we are concerned about the political-economic implications of an economy in which governments control all capital, perhaps we are concerned about the macroeconomic impact of a savings glut. In any case, the bottom constraint is needed by VIKAASA which requires a compact constraint set. Given that constraint, a low debt economy with high taxation rates and high capital would quickly accumulate excess savings.

Our viability kernels have provided us with three conclusions. First, we find a significant tradeoff between equality and efficiency: both the viability kernels and their projections shrunk as we lowered a relative factor share constraint. Second, countries with little capital may be unable to control inequality while also meeting their other objectives. Third, the tradeoff between equality and efficiency does not become more stark in high debt countries – in fact, the high tax rates those countries require will provide them with low inequality anyway.

VI Stabilising paths

Section V discussed viability kernels, the states from which an economy can be controlled to remain within our constraints K indefinitely. These states are not necessarily stable – many will have to be controlled with changing tax rates to remain in K. That will change their χ. While the constrained kernels demonstrate that some states can be controlled while retaining low χ, it was unclear whether χ typically converges as an economy is stabilised. In this section we ask whether it does.

VIKAASA confirms viability of a point by finding a path emanating from that point which leads the economy to a near-steady state. These paths are not unique, and strategies that generate these paths are not the only viable strategies. Nevertheless the patterns resulting from these paths can be illuminating.

We first compare the 'final' values of χ with those 'initial' values shown in Figure 3. Figure 7 depicts the distributions of χ across all viable states. The first panel depicts these states' initial χ, the second depicts their χ once they have been stabilised. Both panels take states from the $\chi \geq 0$ kernel, and allow any viable value of $B \in [-1, 3.5]$.

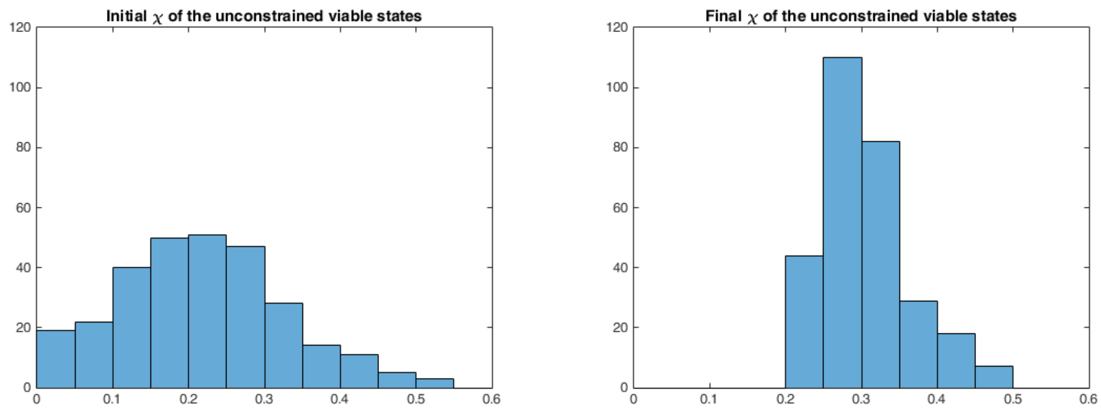

Figure 7: Final and initial χ distributions

As can be seen, the stabilised states of χ are less widely distributed, and in particular there are no stable states with extremely low inequality. If the stable states which correspond to the paths VIKAASA has found are representative of all possible stable states then extremely low inequality is unsustainable.

The next figures show a sample of time profiles of viable evolutions for unconstrained χ – Figure 8 – and $\chi \leq 0.4$ – Figure 9.[15] The selected sample paths are those that start far from a near-steady state. They are sufficiently long for us to see how near-steadiness is achieved.

Strikingly, the figures differ very little. This is because the low capital states – viable without the χ constraint but non-viable with the χ constraint – were already near-stable and thus were not selected for Figure 8.

The time profiles have been collected into three groups: low initial tax rate (black lines), medium initial tax rate (pink lines) and high initial tax rate (red lines). The high tax category tends to have moderate to high relative factor share and the lowest output. This category tends to start with low to medium capital. The time profiles with low initial tax rates (black lines) tend to have low

[15]The graph names are self explaining apart from *velocity* which is the Euclidean norm of each state variable velocity at a point, $\sqrt{\dot{k}^2 + \dot{c} + \dot{B} + \dot{\tau}^2}$. Near-steadiness is achieved when this norm is less than a tolerance parameter.

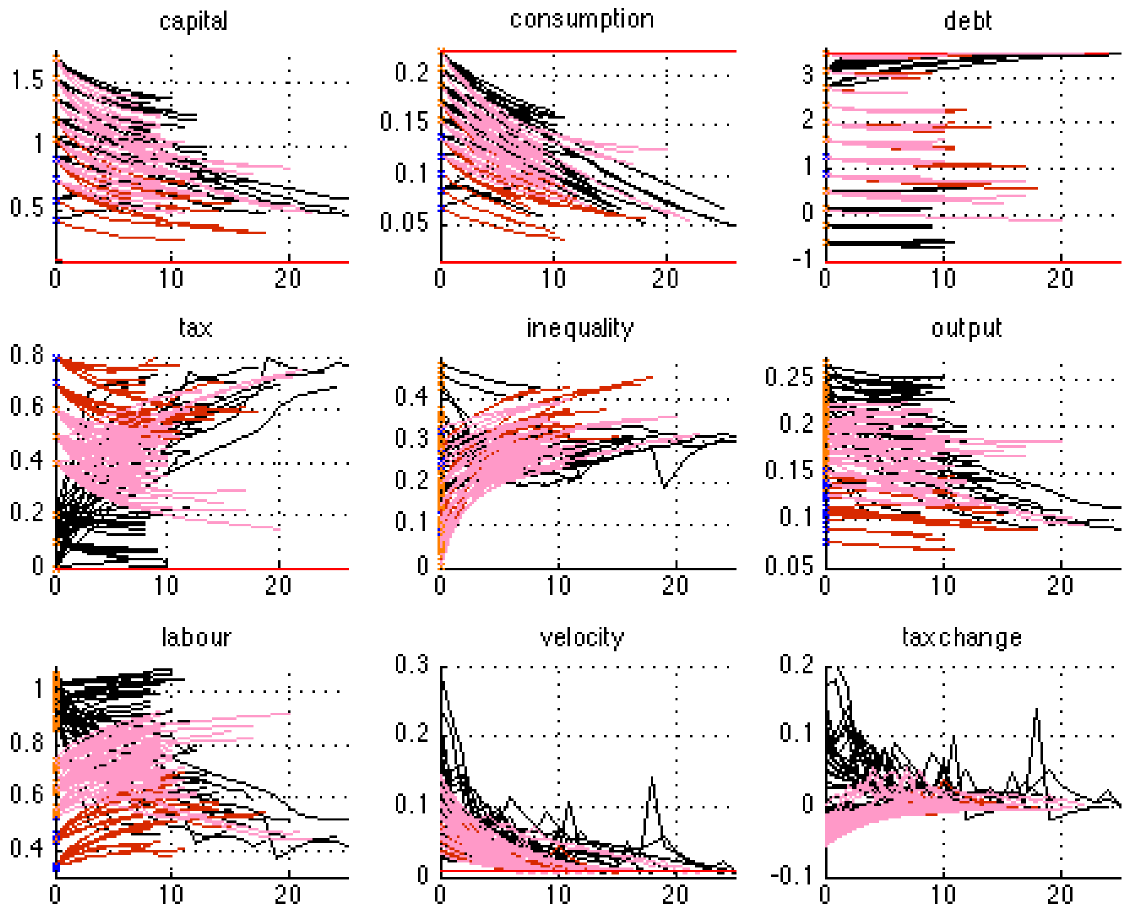

Figure 8: Viable evolutions' time profiles for unconstrained χ

inequality and high output. If initially high taxes (red lines) are to produce viable evolutions with constrained inequality, taxes need be lowered. The medium-high and high taxation paths require labour increases for viability.

Our model has diminishing returns to capital and constant total factor productivity, and so it is unsurprising that few of the paths have growing output. Those which do grow tend to have high debt, tax decreases and labour supply increases. As we mentioned earlier, a longer-run relationship between output and inequality – such as that captured by the Kuznets curve – would require a model designed for a longer time frame.

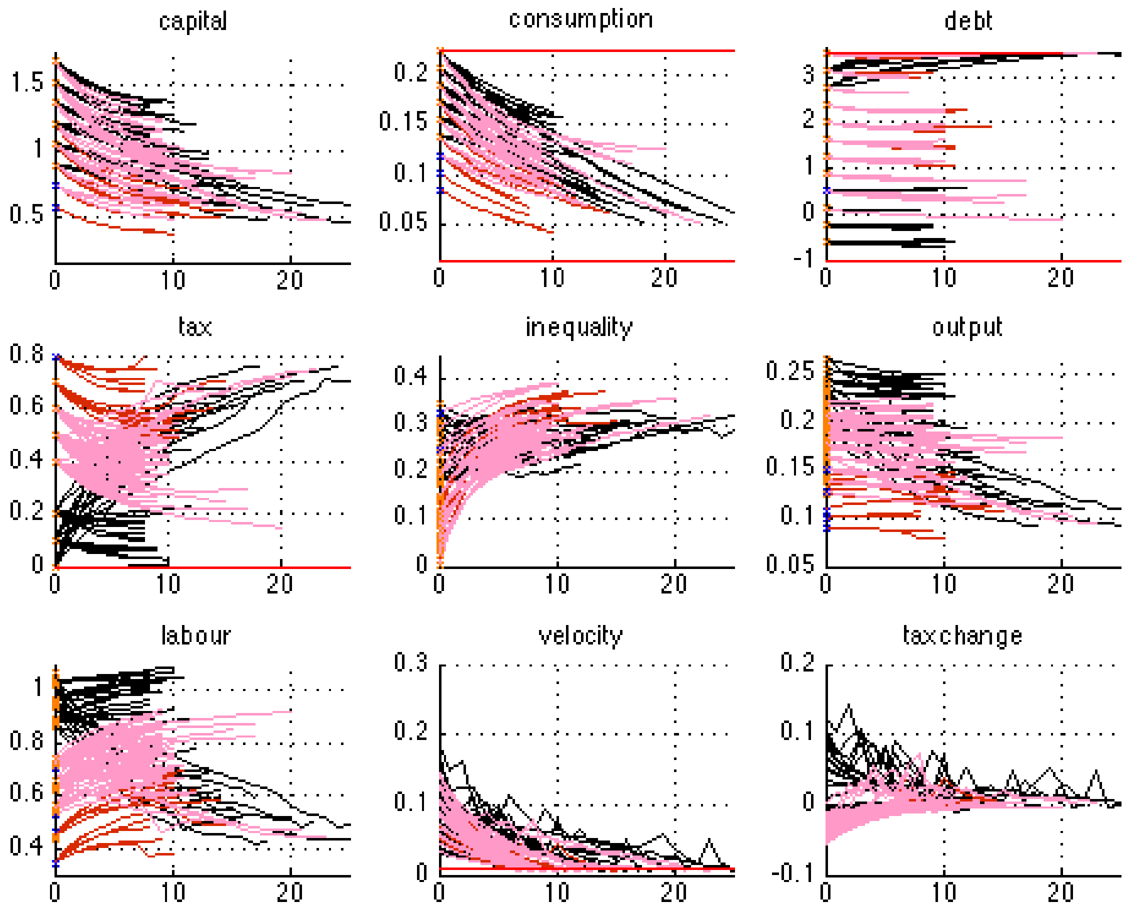

Figure 9: Viable evolutions' time profiles for $\chi \leq 0.4$

Figures 8 and 9 invite little speculation about the relationship between inequality and output: our viable evolutions stabilise at a wide range of steady states which nonetheless have very similar relative factor shares. Nonetheless, that many distinct steady state levels of output can generate similar levels of inequality does have important policy implications. It suggests that the tradeoff between equality and efficiency may be somewhat less sharp than Section V suggested: those states with the lowest inequality were unstable, and while some high-inequality states become non-viable when a $\chi \leq 0.4$ constraint is imposed, the results of this section suggest that those states might have been made to eventually stabilise with $\chi \leq 0.4$ anyway. The next section investigates whether they can.

VII Reducing an economy's relative factor share

In Section V we established the sets of economic states which are sustainable, given different constraints on χ. This section asks how an economy can transition from a high χ state to low χ state.

Figure 10 contains the kernels obtained for $0 \leq \chi$ and $0 \leq \chi \leq 0.4$, marked by the lighter and darker colours respectively. Unsurprisingly, the more constrained kernel is a subset of the less constrained kernel. Both kernels have plenty of viable states with low χ.

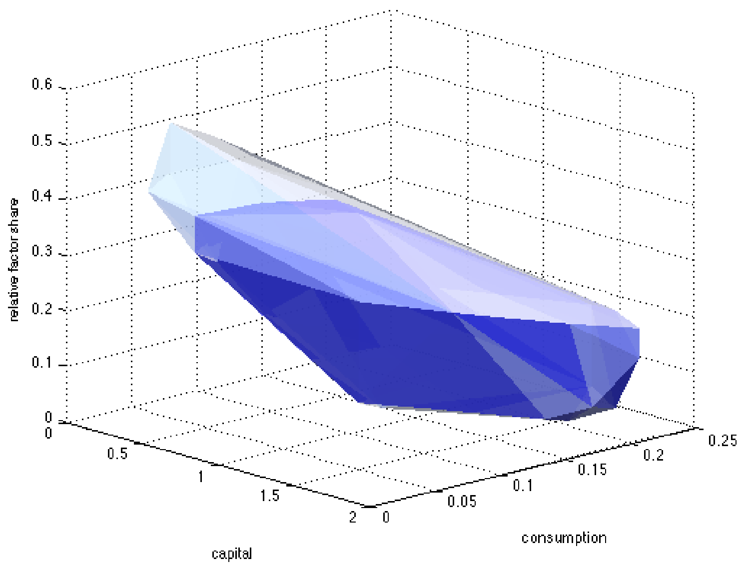

Figure 10: The $0 \leq \chi$ and $0 \leq \chi \leq 0.4$ viability kernels

Consider those states within the bigger kernel with $\chi > 0.4$. They are in the top, light-coloured part of the boulder and have low capital and consumption. We want to establish how a social planner could move this economy from one of these states to one with $\chi \leq 0.4$, sitting in the darker part of the boulder.

Examining the stabilising evolutions which emanate from each viable state reveals the existence of several evolutions which have an initial $\chi > 0.4$ and a stabilised χ below 0.4. Figure 11 shows two of them originating from $\chi = 0.4302$ (red line) and $\chi = 0.4006$ (black line).

Two different slices of the viability kernel are shown in Figure 11. The slice in the left panel has χ on the vertical axis and so is the same as the darker kernel in Figure 10. A different slice of this viability kernel is shown in the right panel, with tax rates on the vertical axis. The lines show two evolutions from $\chi > 0.4$ to $\chi < 0.4$. We can see that the relative factor share diminishes as the taxation rate rises. In each evolution the capital and consumption do not vary substantially.

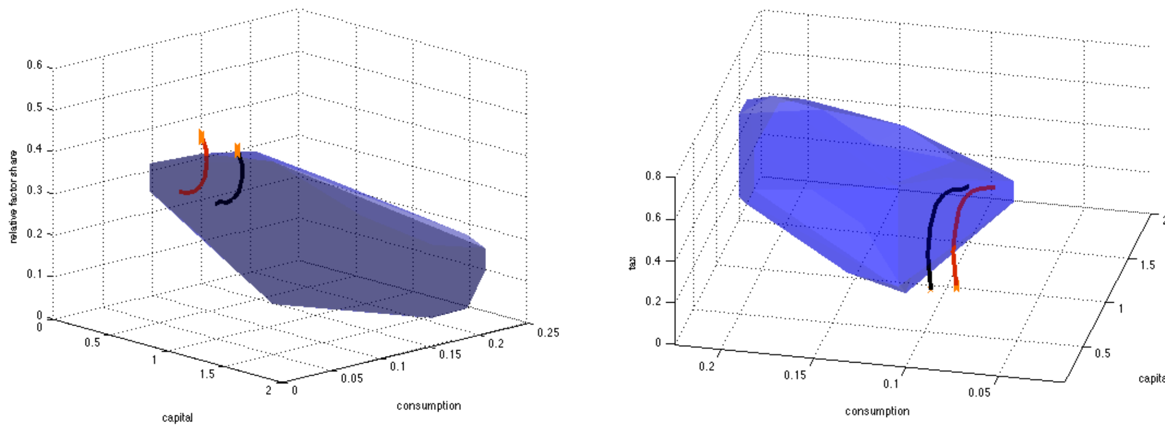

Figure 11: Viable evolutions from a high χ economy to $\chi \leq 0.4$. The inequality transition can be observed in the left panel, the taxation transition in the right.

The evolutions' time profiles are shown in Figure 12. We can see how capital, consumption, debt, tax and χ converge to a near-steady state. This process is a result of controlling the economy by the tax changes shown in the last panel. Near-steadiness is represented by the diminishing velocity in the low middle panel. The profiles show that debt increases fast as tax rates grow while capital and consumption decrese slowly. The tax rates have to grow fast mainly because small tax increments would be too small to keep debt low, when capital decreases.

By applying fast tax increases, the unequal economy with low tax and low debt has transitioned to a more equal economy with high taxes, high debt and low output. Of course we would have preferred to have reached a state with higher output. Finding a strategy that would have taken us to that state would have required solving a crisis-control problem, the general problem in which a state of the dynamic system is outside the viability kernel and the planner wants to steer the

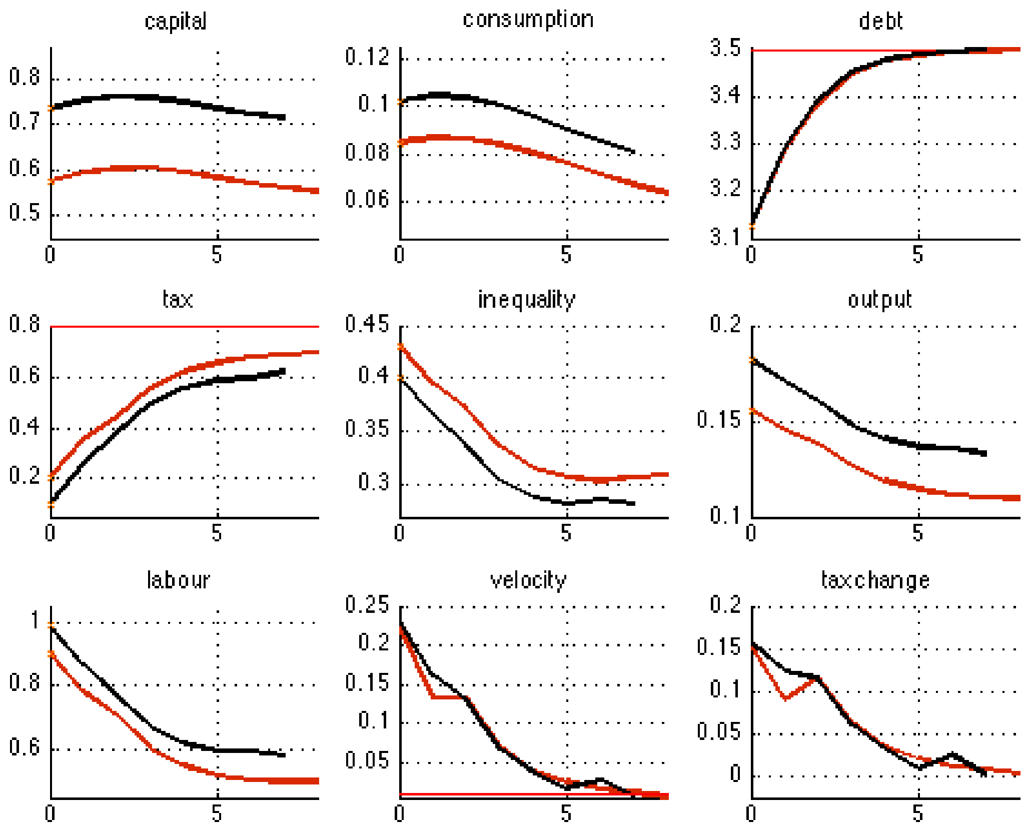

Figure 12: Time profiles of viable evolutions from a high χ economy to a constrained $\chi \in [0,\ 0.4]$

system into the kernel. The state may be outside the kernel because a shock pushed the system outside the kernel, or because the social planner read this paper too late. Doyen and Saint-Pierre (1997) discusses the crisis control problem in greater depth.

VIII Concluding remarks

In the economy we study, concerns about inequality needn't contradict concerns about inefficiency. When we constrain the relative factor share, and by the results of Section II also constrain inequality, we find that many economic states remain viable. However, as the constraint is lowered, the viability kernel shrinks quickly. Whether a policy maker's desires about equality and efficiency conflict will depend on how much inequality they are willing to tolerate. In economies with little capital the tradeoff was particularly stark.

We showed in Section VI that most economies will tend to have similar levels of inequality in the long run, despite remaining quite different in other ways. Countries which initially have low inequality will find their economy becomes more unequal as it stabilises – a result echoing other authors which have found capitalism has a natural tendency toward inequality, such as Piketty (2014).

Viability theory can also tell use how that tendency can be resisted. In Section VII we find that a low capital economy can stabilise itself while reducing inequality, provided it is willing to tolerate low output.

Moving aside from policy advice, if we think of our viability theory in positive terms – as a realistic description of how politicians act – we can produce insightful and accurate explanations of economic phenomena. As shown in Section V, the high taxes and low inequality of Japan, and the low taxes and high inequality of Columbia, are unsurprising given that Japan has a lot of debt and Columbia doesn't.

More important than our specific results is our demonstration that viability theory is a useful approach to understanding the inescapable tradeoffs that our concerns about inequality introduce. In formalising what we consider acceptable we can test whether these considerations are realistic and – if so – how they can be achieved. That will prove immensely valuable to both decision makers and the economists who wish to study them.

Acknowledgment

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A method for finding viability kernels

VIKAASA[16] is a suite of MATLAB® programmes that approximate viability kernels. It can be used either as a set of MATLAB® functions, or via a GUI.[17] The GUI can specify the viability problem, run the kernel approximation algorithms and display the results. A detailed (though somewhat outdated) manual for VIKAASA can be found in Krawczyk and Pharo (2011). The latest version of VIKAASA is available for download at Krawczyk and Pharo (2014).

In this paper, our algorithm solves a truncated optimal stabilisation problem for each element of $K^h \subset K$, a discretisation of K. For each $x^h \in K^h$, VIKAASA assesses whether a dynamic evolution originating at x^h can be controlled to a (nearly) steady state without leaving the constraint set in finite time. Those points that can be brought close enough to such a state are included in the kernel while those that cannot are excluded. This algorithm, the *inclusion* algorithm, will miss viable orbits points that cannot reach a steady state.

[16] See Krawczyk and Pharo (2011), Krawczyk and Pharo (2014), Krawczyk, Pharo, Serea, and Sinclair (2013).

[17] VIKAASA is also compatible with GNU Octave, though its GUI is not.