Soft transactions*

C. Oscar Lau†

30 September, 2015

Abstract

This paper suggests "soft debt" as a social convention that facilitates long-term reciprocal relationships. A player is said to follow a soft debt strategy if his decisions depend on the entire history with his counterpart only through their accrued soft debt balance. Under discrete benefits, there exist equilibria in which the players keep reciprocating as long as the debt balance does not exceed a certain limit. The objective of the paper is to model reciprocal exchanges as close to market transactions as possible. Among other benefits, this approach allows a help seeker to compare the cost of a friend's help with that of buying from the market directly. It may also explain some apparent "perversive" effects of financial incentives observed.

KEYWORDS: Repeated games; Stochastic games; Reciprocity

JEL Classifications: C73

Provisions of goods and services between individuals can be broadly classified into two categories: those involving formal contracts and those not. Formal contracts cover a variety of trades ranging from routine grocery shopping to complicated property transactions. Despite the various degree of complexity, all transactions involve explicit prices, often expressed in monetary terms. They are therefore well studied within the rich and powerful framework of price mechanism in the market environment.

Meanwhile, the reciprocal provisions of goods and services are just as ubiquitous as market transactions, despite being more subtle due to their lack of formal contracts. They

*I am (softly) indebted to Luis Araujo, Thomas Jeitschko and Arijit Mukherjee for their comments and suggestions. I also thank the seminar participants at Michigan State University, Massey University, Victoria University of Wellington, and the Australasian Economic Theory Workshop for their comments. All errors are mine.

†School of Economics and Finance, Massey University, New Zealand. Address: Social Sciences Tower - SST4.09, Massey University, Palmerston North 4442, New Zealand. Phone: +64 6 356 9099 extn 83102. Email: c.l.lau@massey.ac.nz

are essential in our daily lives. Examples include sharing of household duties between family members, collaboration between colleagues, and exchange of favours between friends. Moreover, their significance extends beyond interpersonal relationships; they are common between firms, organizations and even governments. However, due to the lack of explicit prices, they are not covered under the price mechanism. Instead, since their existence rely on intertemporal incentives maintained under strategic environments, they are modeled within the game theory framework.

On the surface, the disparate research approaches seem to be justified in view of the different natures of the two classes of exchanges. But there may be more similarities between them than the different frameworks suggest. For many goods and services, both channels of acquisitions co-exist. For instance, one may ask her partner to cook dinner as a favour or she may dine out; a person planning to move could ask his friends for help or hire professional movers. The close relationship between the two channels is also illustrated by the shift of importance from personal exchanges to formal transactions as market develops.[1] Indeed, although reciprocal exchanges lack explicit prices, the routine use of words such as "owe," "indebted," "pay," "political capital" even in the absence of formal transactions suggests their resemblance.

In this spirit, this paper attempts to model the informal exchanges as close to their formal counterparts as possible. We depict them as "soft transactions," in contrast to "hard transactions" in formal trades. Specifically, we suggest "soft debt" as a social institution that provides the incentive to reciprocate. The premise is that whenever help is provided, a soft debt is tacitly accrued from the beneficiary (the "soft buyer") to the provider (the "soft seller"). The soft debt is then added to the soft debt balance between them. If both of them follow "soft debt strategies," i.e., they make decisions about offering and soliciting / accepting help based on this balance, then their expected future values of the relationship also depend on the balance. The more one is owed, the more likely that he will be helped and the more valuable the favour is expected to be, and thus the higher his expected future value. This consideration creates intertemporal incentives that promote reciprocity.

The model has two features that set it apart from other models of reciprocity, offering advantages in terms of both realism and application. First, in all existing models of

[1] For examples of empirical evidences and theoretical treatments of the interface between personal and market exchanges, see Yellen (1990), Kranton (1996) and Araujo (2004).

reciprocity (that we are aware of) a player automatically accepts favours whenever offered one. In contrast, in the soft transactions model the beneficiary has to accept the offer before a favour is rendered, just as a buyer needs to agree to purchase a good before it changes hands. This standard assumption of automatic acceptance in existing models may seem self-evident: why would a rational player refuse any benefit? However, the so called "favours" are not actually free, but comes with a (soft) price as in hard transactions (and hence the term "soft buyer"). The very essence of reciprocity lies in the art of deciding not just when to give, but also when to receive. This insight further leans support to the notion of soft debt: the soft buyer is "indebted" to the soft seller.

Regarding realism, the requirement of acceptance before a favour is rendered seem to be well substantiated in the real world. Most people would think before soliciting or accepting favours from acquaintances. Between family members and close friends, the acceptance process may be subtle or even routine, but the parties would still be aware of at least a vague history of favours between them. The hesitation to accept may be exhibited even more strongly in primitive societies, where impersonal exchanges play a key role. For instance, ultimatum games conducted in some small-scale societies found that offers even *exceeding* 50 percent were frequently rejected (Henrich et al. (2001)).

With respect to application, the acceptance requirement enables the buyer to compare the prices of the soft and hard options of acquisitions of goods or services side-by-side when both are available. For instance, the person planning for a move can directly weigh the soft price for his friend's help against the hard price for professional movers, and choose the cheaper alternative. Factors that change the soft price, and hence the demand for help, can be studied in the same fashion as hard transactions in the familiar price theory framework. In contrast, under typical models of reciprocity, the person do not have the option of declining (or not seeking) his friend's help in favour of market solutions.

Moreover, since soft transactions are modeled after mutually beneficial and voluntary hard transactions, there is no need to detect and suspect for "defects" when help is not rendered by the counterpart. A player will sell a favour whenever the soft price he would earn exceeds the cost; conversely, he will buy a favour whenever the soft price he would pay is less than the benefit. Detecting defects would have been difficult because the player may not be able to tell whether the counterpart is actually capable of providing help.

Next, bringing soft and hard transactions together may help explain some apparent anomalies of incentives. For example, Gneezy and Rustichini (2000) report that in a group of child day-care centres in Haifa, Israel, imposing monetary fines for lateness to the parents actually increased tardiness significantly. On the surface, the experiment suggests that financial incentives may carry "perversive" effects. But rather than distorting the financial incentive, perhaps the lesson of the story is simply that analyses based purely on the hard elements are incomplete.[2] The apparent anomaly may be explained by taking both the soft and hard sides into consideration. In the absence of fines, the parents may have limited means to pay for their soft debt resulting from lateness. The introduction of fines "hardens" the transaction, providing a way for the parents to repay the teachers. Indeed in the context of firms, the coexistence of formal (hard) and informal (soft) agreements in repeated interactions within and between firms have been well recognized and studied by the relational contract literature (e.g. Bull (1987), Baker et al. (2002), Levin (2003))).

The second feature of the model is that the sizes of benefit and cost are randomly drawn in each round rather than being fixed, which is the case in most models. (An exception is the models of informal insurance, which will be discussed later in this section.) The randomness of benefits and costs helps make the model more realistic. Due to the personalized nature of the relationships, people often face a new and unique situation each time they meet. Even between close partners there are variations among established routines, therefore the sizes of benefit and cost often vary from time to time rather than remain fixed. Had the situation been the same in each round, the players probably could have negotiated a contract applicable to each interaction instead of relying on reciprocity, since the cost of negotiation can be spread over the numerous interactions. Taking the argument one step further, as we will elaborate in Section 2, if such goods or services are so impersonal such that they can be easily standardized, then most likely they would be mass produced and marketed, instead of being provided through personal exchanges.

Regarding application, allowing random benefits and costs would support a richer variety of results and hence better capture the nuances of long term relationships. For instance, if they are fixed the benefit must be greater than the cost for the relationship to be economically meaningful, implying that all trades must be efficient. But with random benefits and costs, it is possible that some potential trades are inefficient (as in Section 5.2). As another illustration, the randomness of the benefits and costs means that the players need to consider the specific benefit and cost drawn in each round before deciding

[2] See Bénabou and Tirole (2006) for a host of other examples where material incentives lead to "perverse" effects.

whether to offer (seller) or accept (buyer) help. As we will show in Section 5, the limit of soft debt a player allows (or is allowed) depends on how close the relationship is, which in turn is reflected by the distribution of benefits and costs.

The rest of the paper is organized as follows. In Section 1, we review the related literature. In Section 2, we discuss the advantages and disadvantages of soft transactions to the traders as compared to their hard counterparts. Section 3 presents a general set-up for the two-player stochastic game with private information. The seller may or may not be able to help, and the ability to help is her private information. We show that an autarky equilibrium always exists, but the first best outcome (i.e., trade occurs if and only if the benefit exceeds the cost) can never be achieved.

Section 4 applies the notion of soft debt to the model by defining the soft debt strategy and the soft debt equilibrium, following by a general analysis. Section 5 shows that under discrete benefits, soft debt equilibria with trades could be sustained when the players follow the "debt limit strategy", a particular form of soft debt strategy whereby players buy and sell as long as the debt balance is within a certain limit. In one version of the model, the cost-to-benefit ratio is fixed. In another version, the cost associated with each benefit is random, where he cost may exceed the benefit and its realized value is observable only to the seller. Although the first best allocation cannot be attained, all trades that do occur are efficient. Section 6 concludes.

1 Related Literature

There is a vast literature on cooperation in repeated games. Typically, in each stage the players have the same set of actions available to them and they choose their actions simultaneously, for example in the form of the prisoner's dilemma game. In reality, however, reciprocity often involves unilateral transfer of favour from one player to another (in expectation of return of favour in the future) in each encounter, rather than bartering for mutual benefits. But as Compte and Postlewaite (2015) points out, the choice to model simultaneous plays is "not based on a concern for realism, but for analytical convenience."

Our choice to model the repetition based on one-sided stage benefit (i.e. only one player could receive benefit in each stage) is necessitated by the premise of the paper. The main objective of the paper is to draw an analogy between reciprocity and hard transactions, and hence enabling the application of the price theory to the former. Hard transactions, except for barters, involve unilateral provision of goods or services, settled by the receiver with money or debt. In soft transactions, it is the same one-way provision that assigns the players as soft buyer or soft seller, allowing them to calculate the soft prices and hence accept or reject trades. Mutually beneficial stage games are analogous to bartering in the sense that all parties could potentially receive real benefits simultaneously. Given the rarity of barters and mutually beneficial plays, our choice of one-sided stage benefit model follows naturally.[3]

One-sided stage benefit models have been studied in a number of "favour trading" papers. However, none of them has any of the two key features of our model, namely the acceptance requirement and the randomness of the sizes of benefits and costs. Möbius (2001) shows that favour trading could be maintained between two players if they grant favours as long as the net balance of favours granted is below a certain number (although the main focus of the paper is on indirect favours in groups). The debt limit strategy that we will present resembles this mechanism. Hauser and Hopenhayn (2008) demonstrates that higher payoffs can be achieved by relaxing the exchange rate between current and future favours and allowing the balance of favours to appreciate or depreciate. Independently, Abdulkadiroglu and Bagwell (2013) also find that declining size of favour owed can improve efficiency in their repeated trust games with private information. Nayyar (2009) presents a discrete time version of Hauser and Hopenhayn's model and allows the opportunities to offer help to occur at different rates for the two players. Kalla (2010) introduces two variants of the model: one allows incomplete information on other player's discount factor players, the other features concave utility functions instead of being risk neutral, hence favour trading is considered a form of insurance. In a related topic, Leo (2015) shows that for tasks involving two individuals taking turns in exerting efforts with stochastic private costs, efficiency can be improved by flexible turn-taking arrangement.

Also closely related is the literature on informal insurance (e.g. Kimball (1988), Coate and Ravallion (1993), Kocherlakota (1996), Ligon et al. (2002)). These papers study how income shock-prone households, particularly those in agrarian economies, maintain mutual insurance arrangements (either explicitly or tacitly) without commitment. Informal insurance can be seen as a form of favour trading, with particular focus on mitigation of risk. Typically, an arrangement specifies for each state or each possible realization of

[3] As Graeber (2011) argues based on anthropological evidences, barter economies pretty much never existed even in primitive societies, in contrast to what economists have long imagined.

history a transfer or loan between two risk-averse households. The implementation of informal insurance relies on choosing certain punishment mechanism. Usually the models assume the most severe punishment of expulsion (i.e. grim trigger strategy), whereby the players follow the insurance arrangement until any of them violate the arrangement, when the players will revert to autarky thereafter. In contrast, the soft transactions model does not require punishment rules; instead it needs to set the rule of how the soft debt is calculated.

This paper is also related to the broader literature on reciprocity. The present model, like many others, takes self-interest maximization as individuals' only objective, even portraying intimate relationships as long series of transactions. But undeniably other regards such as kindness, fairness, and retaliation, play an important role in real human relationships. [4] Also, as pioneered by Trivers (1971) and Axelrod and Hamilton (1981), evolution provides a theoretical basis of reciprocal altruism. Nevertheless, despite the incompleteness of the *homo economicus* paradigm, in long-term relationships there still often exist some degree of calculations based on give-and-take consideration. This paper focuses on the strategic aspects of reciprocal relationships, which are analogous to hard transactions.

2 Soft vs Hard Transactions

As explained in the introduction, one advantage the present model is that it allows direct comparison between acquisitions through hard transaction and soft transaction when both are available. The availability of both channels begs the question of which one is chosen over the other under what situations. To answer this question, we need to compare the advantages and disadvantages to the traders under the two modes of exchanges.

The key advantage of soft transactions over the hard alternatives is their saving in transaction costs. By their very nature, soft transactions involve no (or minimal) negotiation of price and conditions. The absence of formal contracts also means no formal record keeping and no taxes. On the other hand, the lack of formal enforcement mechanism means that they are harder to start between strangers.

[4] See for example Andreoni and Miller (1993), Fehr and Schmidt (1999), Fehr and Gachter (2000), Bowles and Gintis (2002), Falk and Fischbacher (2006) and the empirical evidences cited by them.

It is therefore unsurprising that soft transactions are preferred when the needs are personal and specific, for which customization would be valuable but contracting would be costly. They are well suited for interactions between acquaintances, especially when repeated interactions would further lower the costs as players learn more about each other's tastes and habits.

Also, since the favours are often non-standardized, they tend to be personal services rather than tangible goods. In contrast, when the need is general and standardized, hard transactions gain the advantage by exploiting the economies of scale through mass production and routine transactions.[5] This characteristic of soft transactions underpins the assumption of randomness of benefits and costs in our model.

A second factor that distinguishes between the two alternatives concerns the medium of exchange. Hard transactions enjoy the benefits of having money as the medium of exchange, which facilitates impersonal trades. On the other hand, soft transactions are essentially tacit exchanges that occur over time, which require each party to have something that (i) the other wants and (ii) he can provide at an acceptable cost. Even if multilateral favour trading is allowed in a group setting through indirect favours, as in Möbius (2001), trading in soft debts still cannot match the flexibility and convenience of trading in money. Therefore soft transactions are more viable only when there are common interests between the parties.

As a result of these two factors, soft transactions are pervasive everywhere from the family to the neighborhood to the workplace.[6] Examples include spouses sharing household duties; neighbors trading favours such as baby-sitting and house-sitting; research collaborators taking turns in contributing to their projects. The reliance on personal contacts also suggests that soft transactions play a particularly strong role in the social fabric of developing countries.As mentioned in the introduction, even firms (especially those with specific needs) could exhibit elements of soft transactions in the form of relational contracts.

Soft transactions may also be advantageous in situations where silent mutual understanding is preferred to explicit agreement (e.g. tacit collusion between businesses, po-

[5]Economists have long recognized the differences between personal interactions between acquaintances and anonymous market transactions. Adam Smith examined the two types of social interactions in *The Theory of Moral Sentiments* and *The Wealth of Nations* respectively.

[6]Although a marriage is often accompanied by a contract, the "contract" is probably far too vague to make the marriage a hard transaction.

litical deals between governments), or where hard transactions are downright illegal or considered immoral (e.g. corruption). In some other cases, hard transactions are simply unavailable for legal and ethical reasons. For instance, researchers can contract out tasks only to a certain extent.

Soft transactions could also play a role in property rights allocation problems. For example, although auctioning would be an efficient way for a family to allocating the right to choose TV programs, it is seldom adopted in reality. Instead family members make compromises without negotiating explicit terms of compensation to the conceder, who nevertheless expects to be compensated (or have some of his soft debt offset). The same kind of tacit give-and-take reciprocity is as well practiced between friends, neighbors and coworkers, etc.

3 General Model Set-up

In this section we present a general set-up for a two-player repeated game with two key characteristics: requirement of acceptance before granting favour; and randomness of benefits and costs.

Before presenting the formal model, to fix ideas, think of two friends, Peter and Mary, who help each other from time to time. Suppose Peter needs to move house and he would benefit from Mary's help. Peter is then the soft buyer and Marty the soft seller for this time. Mary may or may not be able to help, depending on whether she will be free. If Mary does not offer help, it could be either because she is genuinely unavailable or she does not find it profitable to sell. Peter cannot distinguish between the two possibilities.

Unlike typical repeated games, the rendering of help requires the consent of not just the provider (the seller), but also the beneficiary (the buyer). Therefore help is rendered only if Mary offers and Peter asks for help.

3.1 Game structure

Two players play an infinitely repeated game, starting in period 0. At the beginning of period t, a random draw is made to determine (i) the role of the players, (ii) whether the seller is capable of helping, and (iii) the benefit and cost if help is rendered. Specifically,

(i) One player is randomly assigned the role of the (soft) buyer, who develops a need and will benefit from the other player's help, if rendered. The other player is assigned as the (soft) seller. The probability of being assigned as the buyer is time-invariant. Let $r_t \in \{1,2\}$ denotes the role assignment, where $r_t = i$ if player i is the buyer. Also, let $\beta_i \equiv Pr(r_t = i)$ be the probability of player i being assigned as the buyer, so that $\beta_1 + \beta_2 = 1$.

(ii) The seller observes the buyer's need and may or may not be able to help. Her ability to help is randomly drawn and is privately observed. Player i's ability to help is denoted by $a_{it} \in \{1,0\}$, where $a_{it} = 1$ if capable and $a_{it} = 0$ otherwise. If assigned as the seller, player i is able to help with probability $\pi_i \equiv Pr(a_{it} = 1 \mid r_t \neq i)$, which is fixed across time. The buyer cannot help himself: $a_{it} = 0$ if $r_t = i$.

(iii) If help is rendered, the buyer (suppose player i) will receive a benefit of $b_{it} > 0$ and the (capable) seller (player j) will incur a cost of $c_{jt} > 0$, drawn from the joint cumulative distribution function $\xi_{ij}(b_{it}, c_{jt})$ for $\widetilde{b}_{it}, \widetilde{c}_{jt}$, otherwise both players will receive zero payoff. The joint distribution ξ_{ij} is independent and identically distributed (i.i.d.) across time.[7]

The probabilities β_i, π_i and the joint cumulative distribution ξ_{ij} for $i \neq j$ are common knowledge. The information (r_t, b_{it}, c_{jt}) is revealed to both players once drawn.[8] The ability to help a_{it} remains the player i's private information forever. (Obviously the buyer can deduce that the seller is capable if help is rendered, but he cannot if otherwise.)

Then the buyer and the capable seller simultaneously decide whether to buy and sell respectively. Let $d_{it} \in \{1,0\}$ be the decision of player i, where 1 denotes to trade and 0 not to trade. Help is rendered if and only if *both* the buyer buys and the seller sells. After the favour, if any, is rendered, the game proceeds to the next period. The random variables are summarized in Table 1.

The players are risk neutral with discount factor δ_1 and δ_2, which are common knowledge.

Information Structure

Both players observe the complete history of roles, benefits, costs and decisions, but the seller's ability to help is only observed by the seller, therefore the game belongs to the

[7]In stochastic games, usually state variables are used to capture the changes in stage games, as in Maskin and Tirole (2001) and Compte and Postlewaite (2015). However, considering the number of random variables and their i.i.d. distributions in our model, it would be more straightforward to define the game without resorting to states.

[8]As a variant, in the model in Section 5.2, the cost of selling is observed only by the seller.

Role (who is buyer)	Ability to help (private info; 1=capable)		Decision (1=to trade)		Benefit and cost			
r_t	a_{1t}	a_{2t}	d_{1t}	d_{2t}	b_{1t}	c_{1t}	b_{2t}	c_{2t}
1 (β_1)	0	1 (π_2)	1	1	>0 (ξ_{12})	0	0	>0 (ξ_{12})
			otherwise		0	0	0	0
		0 ($1-\pi_2$)	1 or 0	0	0	0	0	0
2 (β_2)	1 (π_1)	0	1	1	0	>0 (ξ_{21})	>0 (ξ_{21})	0
			otherwise		0	0	0	0
	0 ($1-\pi_1$)		0	1 or 0	0	0	0	0

Table 1: Summary of variables (probability / distribution in parentheses)

class of stochastic game with private monitoring (Dutta (1995), Mertens (2002)). [9]

The public information for period t is $y'_t \equiv (r_t, b_{1t}, b_{2t}, c_{1t}, c_{2t}) \in Y'$ where $Y' \equiv \{1,2\} \times \mathbb{R}^4_+$ before the decisions to trade, and $y_t \equiv (y'_t, d_{1t}, d_{2t}) \in Y$ where $Y \equiv Y' \times \{1,0\}^2$ after. Public history up to period t is then $h_t = (y_0, y_1, ..., y_{t-1}) \in Y^t$ for $t \geq 1$, $h_0 = \emptyset$. The set of public histories is

$$\mathscr{H} \equiv \cup_{t=0}^{\infty} Y^t$$

where we set $Y^o \equiv \emptyset$.

Player i's private information for period t is a_{it}. His history h_{it} includes both public and private history: $h_{it} = (y_0, a_{i0}, y_1, a_{i1}, ..., y_{t-1}, a_{i,t-1})$. The set of possible histories for both player is

$$\mathscr{H}^{pvt} \equiv \cup_{t=0}^{\infty} (Y \times \{1,0\})^t$$

Player i's information in period t before making decision includes his history plus current public and private information (h_{it}, y'_t, a_{it}).

3.2 Strategies and Equilibria

The solution concept we will use is the *perfect public equilibrium* (PPE), following Fudenberg et al. (1994). Player i's strategy is public if in each period t, it depends only on the public history h_t but not on his private history $(a_{i0}, a_{i1}, ..., a_{i,t-1})$. In terms of our model, the player's trade decision in period t depends on (h_t, y'_t, a_{it}) only. Formally, a

[9] Private monitoring means that some signals about past actions are observed by some players but not others. In this model, past actions (decisions to trade) are public information, but seller's ability to help is observed by the seller only.

pure public strategy for player i is a mapping from the set of public histories combined with his current (public and private) information into the set of decision to trade:

$$\sigma_i : \mathscr{H} \times Y' \times \{1, 0\} \to \{1, 0\}$$

With a profile of public strategies $\sigma = (\sigma_1, \sigma_2)$, player i's expected payoff is

$$U_i(\sigma) = E^{\sigma} \sum_{t=0}^{\infty} \delta_i^t \left(\widetilde{b}_{it} - \widetilde{c}_{it}\right) \tag{1}$$

where the expectation is taken with respect to the probability measure over the set of outcomes induced by σ and the probability distributions.

A PPE is a profile of public strategies that, beginning in period t and given any pubic history h_t, form a Nash equilibrium from that point on.

We now discuss some basic properties of the equilibria.

Definition 1. The *first best allocation* is attained when help is rendered iff the benefit exceeds the cost and the seller is able to help.

For the general model, the following observations can be made (all proofs are relegated to the Appendix).

Proposition 1. *(i) An autarky equilibrium always exists. (ii) The first best allocation can never be achieved in a PPE.*

While part (i) of the proposition confirms that an equilibrium always exists, part (ii) proclaims that inefficiency is inevitable in the equilibria.

4 Soft Debt Model

To apply the notion of soft debt to the general model in the last section, we focus on strategies that prescribe actions based on the soft debt balance.

4.1 Definitions

In each period, if help is rendered, then the buyer accrues a soft debt of $d(b, c)$ to the seller, where b is the benefit to the buyer and c is the cost to the seller. Assume d is invariant across time. Also assume $d > 0$ for all b and c; and d is increasing in both b and c. Then the soft debt holding can be defined as follows:

Definition 2. Player i's *soft debt holding* (i.e. net soft debt balance owed by player j to him) at the beginning of period τ is

$$D_{i\tau} = \sum_{t=1}^{\tau-1} \left[d\left(b_{jt}, c_{it}\right) - d\left(b_{it}, c_{jt}\right) \right]$$

Obviously, $D_{1\tau} = -D_{2\tau}$ and $D_{i0} = 0$. Note also that for simplicity no "interest" (positive or negative) is accrued on the debt.[10]

For each player, let $\mathcal{A} \equiv \{1, 2\} \times \{1, 0\}$ be the set of current role assignments for r_t and ability a_{it}.

Definition 3. A *soft debt strategy* for player i is a pure public strategy mapping from the product set of the sets of current roles, benefit to the buyer, cost to the seller, and soft debt holding into the set of actions:

$$\sigma_i : \mathcal{A} \times \mathbb{R}_+^2 \times \mathbb{R} \to \{1, 0\}$$

Note that σ_i is time independent (i.e. stationary). The player's decisions depend on the entire history of past actions and information only through the current debt balance. That is, the whole history is condensed in the debt balance as far as the player's strategy is concerned.

Definition 4. A *soft debt equilibrium* is a stationary PPE in which both players' follow soft debt strategies.

A *soft transaction* is a trade that occurs (i.e. help is rendered) in a soft debt equilibrium.

Instead of the soft debt, other statistics could have been used in the equilibrium. For instance, the players could have based their decisions on the net number of times of receiving help in the past, as in Möbius (2001). However, the incidences of past favours do not reflect their magnitudes. As discussed in the introduction, a major characteristic of soft transactions, due to its personalized nature, is the variability of the favours. As a dramatic example, a small favour such as baby-sitting is not comparable to a large one such as kidney donation. In contrast, the soft debt is able to summarize the full magnitude

[10] Unlike the nominal interest rate in hard transactions, the interest rate here can be negative, which can reflect fading of memories.

of past favours, while the general functional form $d(b,c)$ provides flexibility as to how the benefit and cost constitute the debt.

4.2 Analysis

In general, the expected future value of the relationship to player i is given by (1).

Meanwhile, along a soft debt equilibrium path, at the beginning of each period (before drawing of the role assignment, seller capability, benefit, and cost) the expected future value depends entirely on the soft debt balance. We can now define soft prices based on these expected future values along the equilibrium path.

Definition 5. Let $V_1(D_1)$ and $V_2(D_2)$ be the players' expected future value in a soft debt equilibrium where D_1 and D_2 $(= -D_1)$ are their respective current debt holdings. Suppose player i is assigned as the buyer and the benefit b_i and cost c_j are drawn, then his *soft buying price* is

$$p_i^{\text{buy}} \equiv \delta_i \left[V_i(D_i) - V_i(D_i - d(b_i, c_j))\right]$$

and as the seller player j's *soft selling price* is

$$p_j^{\text{sell}} \equiv \delta_j \left[V_j(D_j + d(b_i, c_j)) - V_j(D_j)\right]$$

The functions V_1 and V_2 are stationary because the probability distributions for role assignment, benefits and costs of helping are all i.i.d. across time. The soft price is the increase (for the seller) or decrease (for the buyer) in the expected future value of the relationship resulting from the change in the debt balance. Unlike hard prices, the soft price paid by the buyer and that received by the seller can be different. The divergence exists because in soft transactions, there is no money or other exchange media that defines a single price. This means even if the benefit exceeds the cost, mutually beneficial trade may or may not occur. This limitation may offer an explanation for the development of hard transactions to fill in the gap.

Player i will buy iff the incentive compatibility (IC) condition is met: [11]

$$\text{IC (buyer): } b_i > p_i^{\text{buy}}$$

Similarly, player j, if capable, will sell iff:

$$\text{IC (seller): } c_j < p_j^{\text{sell}}$$

Denote by $H_i(D)$ the event that player i is helped given that he is the buyer with debt holding D and the seller is capable, then:

$$\Pr(H_i(D)) = \Pr\left(\widetilde{b}_i > \widetilde{p}_i^{\text{buy}}(D) \text{ and } \widetilde{c}_j < \widetilde{p}_j^{\text{sell}}(-D)\right)$$

where

$$\widetilde{p}_i^{\text{buy}}(D) \equiv \delta_i\left[V_i(D) - V_i\left(D - d\left(\widetilde{b}_i, \widetilde{c}_j\right)\right)\right]$$

and

$$\widetilde{p}_j^{\text{sell}}(-D) \equiv \delta_j\left[V_j\left(-D + d\left(\widetilde{b}_i, \widetilde{c}_j\right)\right) - V_j(-D)\right]$$

are the random buying and selling prices respectively.

In general, the value function is composed of four components that correspond to the following scenarios: player i is the buyer and he is helped; he is the buyer but is not helped; player j is the seller and she is helped; she is the seller but is not helped. The value function of player i at the beginning of a period can thus be formulated recursively:

$$\begin{array}{rll} & V_i(D) & \\ = & \beta_i & \left\{\begin{array}{ll} & \pi_j \Pr(H_i(D))\, E\left[\widetilde{b}_i + \delta_i V_i\left(D - d\left(\widetilde{b}_i, \widetilde{c}_j\right)\right) \mid H_i(D)\right] \\ + & (1 - \pi_j \Pr(H_i(D)))\, \delta_i V_i(D) \end{array}\right\} \\ & +\beta_j & \left\{\begin{array}{ll} & \pi_i \Pr(H_j(-D))\, E\left[-\widetilde{c}_i + \delta_i V_i\left(D + d\left(\widetilde{b}_j, \widetilde{c}_i\right)\right) \mid H_j(-D)\right] \\ + & (1 - \pi_i \Pr(H_j(-D)))\, \delta_i V_i(D) \end{array}\right\} \end{array}$$

We further simplify the notations by defining the following conditional probability and conditional mean benefit and cost:

[11] There is no need to check the individual rationality (IR) conditions because the "worst" expected future value cannot fall below zero (autarky). Also assume that the players do not buy or sell if they are indifferent.

$P_i(D) \equiv \Pr(H_i(D))$, $\bar{b}_i(D) \equiv E\left(\widetilde{b}_i \mid H_i(D)\right)$, $\bar{c}_i(-D) \equiv E(\widetilde{c}_i \mid H_j(-D))$. Then

$$\begin{array}{rcl} & V_i(D) & \\ = & \beta_i & \left\{ \begin{array}{rl} & \pi_j P_i(D)\left[\bar{b}_i(D) + \delta_i E\left[V_i\left(D - d\left(\widetilde{b}_i, \widetilde{c}_j\right)\right) \mid H_i(D)\right]\right] \\ + & (1-\pi_j P_i(D))\,\delta_i V_i(D) \end{array} \right\} \\ & +\beta_j & \left\{ \begin{array}{rl} & \pi_i P_j(-D)\left[-\bar{c}_i(-D) + \delta_i E\left[V_i\left(D + d\left(\widetilde{b}_j, \widetilde{c}_i\right)\right) \mid H_j(-D)\right]\right] \\ + & (1-\pi_i P_j(-D))\,\delta_i V_i(D) \end{array} \right\} \end{array} \quad (2)$$

Upon rearranging terms,

$$V_i(D) = \frac{1}{1-\delta_i} \left\{ \begin{array}{c} \beta_i \left\{ \pi_j P_i(D)\left[\bar{b}_i(D) - \bar{p}_i^{\text{buy}}(D)\right]\right\} \\ +\beta_j \left\{ \pi_i P_j(-D)\left[-\bar{c}_i(-D) + \bar{p}_i^{\text{sell}}(D)\right]\right\} \end{array} \right\} \quad (3)$$

where $\bar{p}_i^{\text{buy}}(D) \equiv E\left[\widetilde{p}_i^{\text{buy}}(D) \mid H_i(D)\right]$ and $\bar{p}_i^{\text{sell}}(D) \equiv E\left[\widetilde{p}_i^{\text{sell}}(D) \mid H_j(-D)\right]$, for $i = 1, 2$.

Some key terms are interpreted as follows:

- $\bar{p}_i^{\text{buy}}(D)$: player i's conditional mean buying price with debt holding D
- $\bar{p}_i^{\text{sell}}(D)$: player i's conditional mean selling price with debt holding D
- $\bar{b}_i(D) - \bar{p}_i^{\text{buy}}(D)$: player i's conditional mean buying surplus with debt holding D
- $-\bar{c}_i(-D) + \bar{p}_i^{\text{sell}}(D)$: player i's conditional mean selling surplus with debt holding D

Therefore, equation (3) shows that a player's expected future value equals the discounted sum of surpluses weighted by the chances of buying and selling. The two conditional mean surpluses must be positive because the player will trade only if it is profitable.

5 Soft Debt Equilibria

Proposition 1 claims that inefficiency is inevitable, the question is to what extent would trades be limited in equilibria. The last section introduces the possibility of using soft debt balance as a mediator for trade. This section provide examples of soft debt equilibria where trades occur as long as the debt balance does not exceed certain debt limits.

5.1 Discrete Benefit Model with Proportional Cost

Solving for equilibria with trade would require some simplifying assumptions to the general model.

Assume $\delta_1 = \delta_2 = \delta$, $\beta_1 = \beta_2 = 1/2$, $\pi_1 = \pi_2 = \pi$, $\left(\widetilde{b}, \widetilde{c}\right)$ are identically distributed for the two players, and focus on symmetric equilibria where V_1 and V_2 coincide. Consider the case where $\widetilde{b}$ follows a uniform discrete distribution, realizing each outcome of $1, .., B$ with probability $1/B$. The cost corresponding to each benefit of b is fixed at αb where$\alpha \in (0, 1)$ is constant, so the first best outcome is for the players to trade in all periods as long as the seller is capable. Although b is not continuous, in practice the benefit can be measured in units as small as one wants.

Suppose $d(\cdot, \cdot)$ is homogenous of degree 1, then $d(b, \alpha b) = b \cdot d(1, \alpha)$, and $D_{i\tau} = \left[\sum_{t=0}^{\tau-1} (b_{jt} - b_{it})\right] \cdot d(1, \alpha)$. Since $d(1, \alpha)$ is constant, we can use the net cumulative benefit (call it the debt holding level) as an index for the debt balance. Denote by V_k the expected future value when the debt holding level is k:

$$V_k \equiv V(k \cdot d(1, \alpha))$$

Define $p_{k+b,k} \equiv \delta(V_{k+b} - V_k)$, $b = 1, .., L - k$ as the soft selling price received by the seller holding debt level k for providing a benefit of b (or equivalently the soft buying price paid by the buyer for a benefit of b when his debt holding is $k + b$). Note that $p_{k+b,k} = \sum_{r=0}^{b-1} p_{k+r+1,k+r}$. That is, the soft price for b units of benefits is the sum of b one-unit soft prices.

Definition 6. A *debt limit strategy* with limit $L > 0$ is a strategy whereby the player buys (if he is drawn as the buyer) and sells (if capable seller) iff debt levels in absolute terms do not exceed L both before and after the transaction.

A *debt limit equilibrium* with limit L is a soft debt equilibrium in which both players follow the debt limit strategy with limit L.

In other words, under the strategy a player will trade as long as (i) he is capable and (ii) after the trade, neither he will owe nor he will be owed more than L. Once the debt level falls outside the limit (which is off equilibrium path), autarky prevails forever. The limit L defines the range of soft debt level that the players engage in. A higher L means more opportunity for soft transactions. Therefore L can be regarded as depth of relationship.

For a strategy profile to constitute an equilibrium, the following IC and IR conditions have to be satisfied:

$$\text{IC (buyer): } p_{k,k-b} < b \text{ if } b = 1, .., L+k$$

$$\text{IC (seller): } p_{k+b,k} > \alpha b \text{ if } b = 1, .., L-k$$

$$\text{IR: } V_k > 0$$

for $k = 0, \pm 1, .., \pm L$.

The IC conditions ensure that it is profitable for the buyer to buy and the seller (if capable) to sell when the resulting debt level falls within the limit. The IR condition guarantees that the player will always stay in the relationship. Note that the IR conditions are interim IR conditions, i.e., they need to be satisfied not only ex ante (before the drawing in each period), but also ex post.

There are a total of $L\,(L+1)$ inequalities in the IC condition for buyer, the same number in the IC condition for seller, and $2L$ in the IR condition.

From Proposition 1 we already know that the first best allocation cannot be attained. The following proposition shows that when the costs are low enough, simple "debt limit strategies" would constitute equilibria with trade.

Proposition 2. *Consider any positive integer $L \leq B/2$. In the discrete benefit model with proportional cost, if*

$$\alpha < \frac{1}{2B\left(\frac{1/\delta - 1}{\pi}\right) + 1} \tag{4}$$

then there exists a debt limit equilibrium with limit L.

The lower B is, the higher the cost ratio α that can be sustained in an equilibrium. Conversely, a high B means an equilibrium is possible only if the α is low. This is because a high B implies a high potential limit L, which means a player may find herself owed a lot from the other. The cost ratio then needs to be low enough to entice the players to engage in the relationship.

The higher δ and π are, the higher α that can be supported. If δ is high, which means the players are patient and meet frequently, then an equilibrium can be sustained even with a high α. On the other hand, a high π means the players are able to help each other with high probability, which again make a high cost ratio affordable. The affordable cost ratio approaches 1 as δ and π both tend to 1.

For deep relationships like those between family members, δ and π are high because they meet frequently and are often able to help each other. Also, α is low because they are familiar with each other's taste, habit, information, etc., which means it takes relatively little effort to help each other. Therefore a high B, and thus L can be supported. Conversely, between strangers δ and π are low and α is high, so only a low L is possible.

It is obvious from Proposition 2 that multiple debt limit equilibria can coexist. As long as the condition on α is satisfied, all debt limit equilibria with limits smaller or equal to $B/2$ exist. In addition there is always the autarky equilibrium. At first glance the equilibrium seems indeterminate. However, a closer examination would reveal that players will always incline to reach the highest equilibrium limit ($B/2$ for even B or $B/2-1$ for odd B) supported by the cost structure. First note the following proposition.

Proposition 3. *Denote by V_k^L the expected future value in the debt limit equilibrium with limit L when the player's debt holding level is k. Then $V_k^{L'} > V_k^L$ iff $L' > L$ for all $k = 0, \pm 1, .., \pm L$.*

The intuition for the proposition is as follows. A higher debt limit would allow more trading opportunities that are not available under a lower limit. For instance, a player with debt holding level k could potentially buy a benefit up to $L' + k$ or sell a benefit up to $L' - k$ under limit L'. The range would be narrower if the limit is L instead. The difference in the scope of trade holds true in every period. Since the trades are mutually beneficial, the expected future value is higher with a higher debt limit.

In any period, the remaining candidates of equilibrium limits are bounded between the highest debt level attained thus far at the lower end, and $B/2$ at the upper end. Now suppose in the current period if the transaction goes through the debt level will exceed the highest attained level but not $B/2$. The buyer in this period would have no reservation in buying. If the seller sells, not only that the buyer will profit from the transaction, the transaction will also signify a higher limit, which means a higher expected future value (as shown in Proposition 3). If the seller does not sell, trade will not occur but there is no harm in trying to buy anyway. Conversely, the same logic applies to the (capable) seller. Being aware of each other's calculation, the players will always agree to trade within the maximum limit $B/2$.

Therefore although multiple equilibria exist, the one with the maximum limit will prevail ultimately.

5.2 Discrete Benefit Model with Random Private Cost

In this section, the discrete benefit model is modified by assuming instead of a fixed cost-to-benefit ratio, that the cost is random and only observable to the seller. In particular, for each benefit $b \in \{1, 2, .., B\}$, the corresponding cost $\widetilde{c}_b$ follows some distribution over $(\underline{c}_b, \overline{c}_b)$ where $0 < \underline{c}_b < \overline{c}_b$. The upper limit $\overline{c}_b$ can exceed b, therefore rendering help can be inefficient. The distribution of $\widetilde{c}_b$ is common knowledge. Let $\widehat{c}_b \equiv E(\widetilde{c}_b)$.

Since the actual cost is unobservable to the buyer, costs do not enter the soft debt formula. The cumulative net benefit is taken as the soft debt balance, i.e., $D_{i\tau} = \left[\sum_{t=0}^{\tau-1}(b_{jt} - b_{it})\right]$. Like before, V_k denotes the expected future value when the soft debt holding is k, and $p_{k+b,k}$ denotes the soft price $p_{k+b,k} \equiv \delta(V_{k+b} - V_k)$.

The IC condition for buyer and the IR condition are the same as in the last subsection. The IC condition for (capable) seller is:

$$\text{IC (seller): } p_{k+b,k} > \overline{c}_b \text{ if } b = 1, .., L-k,\ k = 0, \pm 1, .., \pm L$$

This IC condition ensures that the capable seller is willing to sell even if the highest cost for delivering the benefit is drawn.

Proposition 4. *Consider any positive integer $L \leq B/2$. In the discrete benefit model with random private cost, if*

$$\overline{c}_b < \frac{b\left(L + k + \frac{b+1}{2}\right) + \sum_{r=0}^{b-1} \widehat{c}_{L-k-r}}{2B\left(\frac{1/\delta - 1}{\pi}\right) + 2L + 1} \tag{5}$$

for $b = 1, .., 2L$, $k = -L, -L+1, .., L-b$, then there exists a debt limit equilibrium with limit L.

All transactions that occur under the equilibrium are efficient, i.e. $\overline{c}_b < b$ for all $b \leq 2L$. Moreover, there always exists some distributions of $\{\widetilde{c}_b,\ b = 1, .., 2L\}$ that satisfy (5).

Again L measures the depth of the relationship. The more frequently the players meet (higher δ), or the more likely the seller is able to help (higher π), or the smaller the maximum benefit B (which means a lower potential L), the easier it is for the cost structure to support the equilibrium. On the other hand, the more efficient they are in helping each other (lower $\overline{c}_b$'s in general), the higher L can be sustained. Therefore the debt limit is highest in closely knit groups where members understand each other's needs well, such as the family. At the other extreme, the debt limit would be very low between strangers.

As shown in the proof, the right-hand side of (5) in the proposition is just $p_{k+b,k}$. Therefore (5) is equivalent to IC (seller). To understand the formula for the soft price, it would be easier to start with the soft price for one unit of benefit:

$$p_{k+1,k} = \frac{L + k + 1 + \widehat{c}_{L-k}}{2B\left(\frac{1/\delta - 1}{\pi}\right) + 2L + 1}$$

This is the increase in future value when the debt holding increases from k to $k+1$. The value increase for two reasons. First, the higher debt holding opens up the opportunity to receive a maximum benefit of $L + k + 1$ (which will bring his debt position to the lowest limit). Second, at the same time he owes his counterpart one less (or she owes him one more), so he will not help if the benefit drawn is $L - k$ or more (the $\widehat{c}_{L-k}$ term).

These factors are adjusted by the denominator for realizing these additional values under different possibilities in different future time. The higher B is, the lower the chance that the future benefits and costs fall within the debt limit and hence the less likely that the values can be realized soon. On the other hand, the higher δ is, either because the agents meet more frequently or they are more patient, the higher the values will be. The soft price for b units of benefit $p_{k+b,k}$ is just the summation of b number of one-unit soft price.

As in the last subsection, although multiple equilibrium may exist, the players will gravitate toward the highest limit $L = \frac{B}{2}$. But note one difference between the two models. In for the constant cost ratio model, if the equilibrium with limit $L/2$ exists, then all equilibria with lower limits exist too. In the current model, the existence of an equilibrium does not guarantee that all equilibria with lower limits exist too. Whether the equilibria exist depends on whether (5) is satisfied by the corresponding costs.

6 Conclusion

This paper highlights soft debt as an incentive device that facilitates reciprocity under private information. Players engaged in soft transactions perform profit calculations using soft prices just like in hard transactions. The first best allocation can never be achieved.

Nevertheless, trading within certain debt limits are possible depending on the discount factor, chance of the seller being able to help, and the distribution of benefit and cost.

There are several directions in which the notion of soft transactions can be extended. First, the paper focuses on isolated bilateral relationship and avoids mentioning soft markets. The model assumes the players are randomly matched and only the seller can help the buyer. But often the buyer would be able to choose from different sellers, and vice versa. Just like in the hard market, the soft prices will be the driving force behind the player's choices. In this setting, whether a relationship is exclusive (e.g. marriage) or non-exclusive (e.g. friends) would affect the player's decisions since breaking up an exclusive relationship can be costly. Another possibility is to consider multilateral relationship in a group, where indirect favours could be granted (as in Möbius (2001)).

Second, the scope of soft transactions is inherently limited by the lack of money as a medium of exchange. Unlike hard prices, the soft price paid by the buyer and that received by the seller can be different. The divergence means that mutually beneficial trades may not occur. This limitation may offer an explanation for the rise of hard transactions to fill the gap. This approach may complement the existing literature on the interface between personal and market exchanges (e.g. Kranton (1996), Araujo (2004)).

Third, this paper presumes a simple dichotomy of hard and soft transactions, while in reality there are many "hybrid" transactions. For example, an employment contract contains both the hard (employment contract) and soft (vague duties within "reasonable" bounds). The relational contract literature (e.g. Bull (1987), Levin (2003), Fong and Li (2012)) is pertinent to the subject since it is concerned with combining explicit and implicit incentives in repeated long-term relations. In fact, to the extent that a contract is incomplete, there is always some "softness" in it and hence there is potential for long-term relationship. In purely hard transactions which allow no room for ambiguity, if they ever exist, the parties would simply trade and part. The presence of hybrid transactions may also help explain the puzzle that providing rewards and punishments sometimes has perverse effects.

References

Abdulkadiroglu, A. and K. Bagwell: 2013, 'Trust, Reciprocity, and Favors in Cooperative Relationships'. *American Economic Journal: Microeconomics* **5**(2), 213–59.

Andreoni, J. and J. H. Miller: 1993, 'Rational Cooperation in the Finitely Repeated Prisoner's Dilemma: Experimental Evidence'. *The Economic Journal* **103**(418), pp. 570–585.

Araujo, L.: 2004, 'Social norms and money'. *Journal of Monetary Economics* **51**(2), 241 – 256.

Axelrod, R. and W. Hamilton: 1981, 'The evolution of cooperation'. *Science* **211**(4489), 1390–1396.

Baker, G., R. Gibbons, and K. J. Murphy: 2002, 'Relational Contracts and the Theory of the Firm'. *The Quarterly Journal of Economics* **117**(1), 39–84.

Bénabou, R. and J. Tirole: 2006, 'Incentives and Prosocial Behavior'. *The American Economic Review* **96**(5), 1652–1678.

Bowles, S. and H. Gintis: 2002, 'Social Capital and Community Governance'. *The Economic Journal* **112**(483), F419–F436.

Bull, C.: 1987, 'The Existence of Self-Enforcing Implicit Contracts'. *The Quarterly Journal of Economics* **102**(1), 147–159.

Coate, S. and M. Ravallion: 1993, 'Reciprocity without commitment: Characterization and performance of informal insurance arrangements'. *Journal of Development Economics* **40**(1), 1 – 24.

Compte, O. and A. Postlewaite: 2015, 'Plausible Cooperation'. *PIER Working Paper No. 15-006.*

Dutta, P. K.: 1995, 'A Folk Theorem for Stochastic Games'. *Journal of Economic Theory* **66**(1), 1 – 32.

Falk, A. and U. Fischbacher: 2006, ‘A Theory of Reciprocity’. *Games and Economic Behavior* **54**(2), 293 – 315.

Fehr, E. and S. Gachter: 2000, ‘Fairness and Retaliation: The Economics of Reciprocity’. *The Journal of Economic Perspectives* **14**(3), 159–181.

Fehr, E. and K. M. Schmidt: 1999, ‘A Theory of Fairness, Competition, and Cooperation’. *The Quarterly Journal of Economics* **114**(3), pp. 817–868.

Fong, Y.-F. and J. Li: 2012, ‘Relational Contracts, Efficiency Wages, and Employment Dynamics’. *Working Paper.*

Fudenberg, D., D. Levine, and E. Maskin: 1994, ‘The Folk Theorem with Imperfect Public Information’. *Econometrica* **62**(5), pp. 997–1039.

Gneezy, U. and A. Rustichini: 2000, ‘A Fine Is a Price’. *The Journal of Legal Studies* **29**(1), 1–17.

Graeber, D.: 2011, *Debt: The First 5,000 Years.* Melville House.

Hauser, C. and H. A. Hopenhayn: 2008, ‘Trading Favors: Optimal Exchange and Forgiveness’. *Working Paper No. 88, Collegio Carlo Alberto.*

Henrich, J., R. Boyd, S. Bowles, C. Camerer, E. Fehr, H. Gintis, and R. McElreath: 2001, ‘In Search of Homo Economicus: Behavioral Experiments in 15 Small-Scale Societies’. *The American Economic Review* **91**(2), pp. 73–78.

Kalla, S. J.: 2010, ‘Essays in Favor-trading’. Ph.D. thesis, University of Pennsylvania.

Kimball, M. S.: 1988, ‘Farmers’ Cooperatives as Behavior toward Risk’. *The American Economic Review* **78**(1), pp. 224–232.

Kocherlakota, N. R.: 1996, ‘Implications of Efficient Risk Sharing without Commitment’. *The Review of Economic Studies* **63**(4), pp. 595–609.

Kranton, R. E.: 1996, ‘Reciprocal Exchange: A Self-Sustaining System’. *The American Economic Review* **86**(4), pp. 830–851.

Leo, G. C.: 2015, ‘Taking Turns’. Technical report, Vanderbilt University Working paper.

Levin, J.: 2003, 'Relational Incentive Contracts'. *The American Economic Review* **93**(3), 835–857.

Ligon, E., J. P. Thomas, and T. Worrall: 2002, 'Informal Insurance Arrangements with Limited Commitment: Theory and Evidence from Village Economies'. *The Review of Economic Studies* **69**(1), 209–244.

Maskin, E. and J. Tirole: 2001, 'Markov perfect equilibrium: I. Observable actions'. *Journal of Economic Theory* **100**(2), 191–219.

Möbius, M.: 2001, 'Trading Favors'. *Working Paper, Harvard University*.

Mertens, J.-F.: 2002, 'Chapter 47 Stochastic games'. Vol. 3 of *Handbook of Game Theory with Economic Applications*. Elsevier, pp. 1809 – 1832.

Nayyar, S.: 2009, 'Essays on Repeated Games'. Ph.D. thesis, Princeton University.

Trivers, R. L.: 1971, 'The Evolution of Reciprocal Altruism'. *The Quarterly Review of Biology* **46**(1), pp. 35–57.

Yellen, J. E.: 1990, 'The transformation of the Kalahari! Kung'. *Scientific American* **262**, 96–105.

Appendix

Proof of Proposition 1

Proof. (i) If one player's strategy is to never buy or sell, there will be no trade regardless of the other player's response. Therefore a best response of the the other player is to follow the same strategy. An autarky equilibrium is thus sustained.

(ii) Assume on the contrary that the first best allocation is attained in a PPE. Suppose player 1 deviates from the existing strategy by not selling although he is able to help at a cost less than the benefit to player 2. Since player 2 cannot observe whether he could help, she would continue the existing strategy which prescribe her to help whenever she can at a cost below the benefit. Player 1 therefore can deviate and profit. The same is true for player 2. This means the PPE that supports the first best allocation does not exist. □

Proof of Proposition 2

Proof. We first compute the expected future values and soft prices assuming that both players follows the debt limit strategy. Next by using these results we verify that the incentive compatibility (IC) and individual rationality (IR) conditions hold. Finally, we argue that the debt limit equilibria exist using the one-shot deviation principle.

1. Computation of expected future value and soft prices

Suppose the debt limit has never been exceeded. Starting with a debt level between $-L$ and L in any arbitrary period, since $B \geq 2L$, the debt level after a transaction (if it occurs) can be any integer within the same range. Given that all transactions go through iff the debt level after transaction remains within $[-L, L]$ and the seller is capable, then for $-L \leq k \leq L$,

$$\begin{aligned} V_k = \quad & \frac{1}{2}\left[\frac{\pi}{B}\sum_{b=1}^{L+k}(b+\delta V_{k-b}) + \left(1-\frac{\pi}{B}(L+k)\right)\delta V_k\right] \\ + \quad & \frac{1}{2}\left[\frac{\pi}{B}\sum_{b=1}^{L-k}(-\alpha b+\delta V_{k+b}) + \left(1-\frac{\pi}{B}(L-k)\right)\delta V_k\right] \end{aligned} \tag{6}$$

In the first bracket the player is assigned as the buyer. The summation term in the bracket refers to the benefits and resulting future values if transaction occurs (with his resulting debt holding level falling to $k-1,\ k-2, .., \ -L$), while the next term captures the no transaction case. Similarly, in the second bracket the player is drawn as the seller, with his debt holding level rising to $k+1,\ k+2, .., \ L$ if trade goes through.

Group all V_k terms to the left hand side,

$$2\left[\frac{B(1-\delta)}{\pi} + L\delta\right]V_k = \sum_{b=1}^{L+k}(b+\delta V_{k-b}) + \sum_{b=1}^{L-k}(-\alpha b+\delta V_{k+b}) \tag{7}$$

where $\sum_{l=1}^{0} \equiv 0$.

Iterate k forward to $k+1$, then for $-L-1 \leq k \leq L-1$,

$$2\left[\frac{B(1-\delta)}{\pi} + L\delta\right]V_{k+1} = \sum_{b=1}^{L+k+1}(b+\delta V_{k-b+1}) + \sum_{b=1}^{L-k-1}(-\alpha b+\delta V_{k+b+1}) \tag{8}$$

Subtract (7) from (8), for $-L \leq k \leq L-1$,

$$2\left[\frac{B(1-\delta)}{\pi}+L\delta\right](V_{k+1}-V_k)=(L+k+1)+\delta V_k+\alpha(L-k)-\delta V_{k+1}$$

$$\left[\frac{2B\left(^1/_\delta-1\right)}{\pi}+2L+1\right]\delta(V_{k+1}-V_k)=L+k+1+\alpha(L-k)$$

Recall the definition $p_{k+1,k}\equiv\delta(V_{k+1}-V_k)$,

$$p_{k+1,k}=\frac{L+k+1+\alpha(L-k)}{2B\left(\frac{^1/_\delta-1}{\pi}\right)+2L+1} \tag{9}$$

Since $p_{k+b,k}=\sum_{r=0}^{b-1}p_{k+r+1,k+r}$ for $b=1,..,L-k$,

$$p_{k+b,k}=\frac{L+k+\frac{b+1}{2}+\alpha\left(L-k-\frac{b-1}{2}\right)}{2B\left(\frac{^1/_\delta-1}{\pi}\right)+2L+1} \tag{10}$$

2. Verification of IC and IR

Now we show that under (4) the players indeed find it profitable to trade whenever the debt level will remain within L. The IC and IR conditions are restated below:

$$\text{IC (buyer): } p_{k,k-b}<b \text{ if } b=1,..,L+k$$

$$\text{IC (seller): } p_{k+b,k}>\alpha b \text{ if } b=1,..,L-k$$

$$\text{IR: } V_k>0$$

By (10), IC (seller) can be rewritten as:

$$\alpha<\frac{L+k+\frac{b+1}{2}}{2B\left(\frac{^1/_\delta-1}{\pi}\right)+L+k+\frac{b+1}{2}}$$

Since k is lowest at $k=-L$ and b is lowest at $b=1$, therefore $L+k+\frac{b+1}{2}\geq 1$. By substituting this lowest value in IC (seller), we can obtain a sufficient condition for IC (seller):

$$\alpha<\frac{1}{2B\left(\frac{^1/_\delta-1}{\pi}\right)+1}$$

which is just (4) in the proposition.

To show that IC (buyer) is met, first note that IC (buyer) is equivalent to:

$$p_{k+1,k} < 1 \ , \ k = -L, -L+1, .., L-1$$

(This restated IC (buyer) is obviously a sufficient condition for the original one. For the necessity part, note that if $p_{r+1,r} \geq 1$ for any $r = -L, -L+1, .., L-1$, then the corresponding inequality for $b = 1$ and $k = r+1$ in the original IC (buyer) will be violated.)

Use (9) and rearrange terms, IC (buyer) becomes:

$$\alpha < \frac{2B\left(\frac{1/\delta - 1}{\pi}\right) + L - k}{L - k}$$

which must hold because $\alpha < 1$.

To verify IR, since V_k is increasing in k (as $p_{k+1,k} > 0$), it is sufficient to show that $V_{-L} > 0$.

By (7), when $k = -L$,

$$\begin{aligned}
2\left[\frac{B(1-\delta)}{\pi} + L\delta\right] V_{-L} &= \sum_{b=1}^{2L} \left(-\alpha b + \delta V_{-L+b}\right) \\
&= \sum_{b=1}^{2L} \left(-\alpha b + \delta V_{-L} + p_{-L+b,-L}\right) \\
\frac{2B(1-\delta)}{\pi} V_{-L} &= \sum_{b=1}^{2L} \left(-\alpha b + p_{-L+b,-L}\right)
\end{aligned}$$

which is positive according to IC (seller) for $k = -L$. Therefore IR also holds given (4).

3. Existence of equilibrium

The one-shot deviation principle states that a strategy profile constitutes a subgame perfect equilibrium iff there is no profitable one-shot deviation for any player at any history. Consider three cases. (i) If the debt level has ever exceeded the limit, then the other player will never buy or sell. So there can be no profitable deviations. (ii) If the debt level has never exceeded the limit but will after the transaction, then again the other player will not buy or sell, and there can be no profitable deviations. (iii) If the debt level has never exceeded the limit and will remain so after the transaction, the IC conditions above guarantee that any deviation will be unprofitable.

In conclusion, by the one-shot deviation principle, the debt limit strategies with limit $L \leq {}^{B}/_{2}$ do constitute debt limit equilibria provided that (4) holds. □

Proof of Proposition 3

Proof. Restate (6) for V_k^L:

$$\begin{aligned} V_k^L = & \quad \frac{1}{2}\left[\frac{\pi}{B}\sum_{b=1}^{L+k}\left(b+\delta V_{k-b}^L\right)+\left(1-\frac{\pi}{B}\left(L+k\right)\right)\delta V_k^L\right] \\ & + \quad \frac{1}{2}\left[\frac{\pi}{B}\sum_{b=1}^{L-k}\left(-c_b+\delta V_{k+b}^L\right)+\left(1-\frac{\pi}{B}\left(L-k\right)\right)\delta V_k^L\right] \end{aligned}$$

where $c_b = \alpha b$

Compare it to $V_k^{L'}$, written as:

$$\begin{aligned} V_k^{L'} = & \quad \frac{1}{2}\left\{ \begin{array}{l} \quad \left[\frac{\pi}{B}\sum_{b=1}^{L+k}\left(b+\delta V_{k-b}^{L'}\right)+\left(1-\frac{\pi}{B}\left(L+k\right)\right)\delta V_k^{L'}\right] \\ + \quad \frac{\pi}{B}\sum_{b=L+k+1}^{L'+k}\left(b+\delta V_{k-b}^{L'}-\delta V_k^{L'}\right) \end{array}\right\} \\ & + \quad \frac{1}{2}\left\{ \begin{array}{l} \quad \left[\frac{\pi}{B}\sum_{b=1}^{L-k}\left(-c_b+\delta V_{k+b}^{L'}\right)+\left(1-\frac{\pi}{B}\left(L-k\right)\right)\delta V_k^{L'}\right] \\ + \quad \frac{\pi}{B}\sum_{b=L-k+1}^{L'-k}\left(-c_b+\delta V_{k+b}^{L'}-\delta V_k^{L'}\right) \end{array}\right\} \end{aligned}$$

There are two differences between the formula for V_k^L and that for $V_k^{L'}$. First, for $V_k^{L'}$, there are two extra summation terms outside the brackets. They are summations of surpluses from trades and thus are both positive. Second, the V^L terms in the first formula are replaced by $V^{L'}$terms in the second. But we can expand the V^L and $V^{L'}$ terms again in the same fashion as above. $V^{L'}$ again has two extra positive terms over V^L. Continue the process repeatedly, the difference due to the unexpanded $V^L$and $V^{L'}$terms tend to zero because of discounting. However, $V_k^{L'}$ accumulates two extra positive terms in each iteration, therefore$V_k^{L'} > V_k^L$.

When there arise a period where the drawing reveals that a transaction would push the debt level over k for the first time but remain below k'. Each player will be better-off by trading given the other player will trade too. Their future value for the same debt level will be higher, and they will gain from the transaction. The equilibrium with the highest limit dominates all the rest. □

Proof of Proposition 4

Proof. Like in the proof for Proposition 2, we first compute the expected future values and soft prices assuming that both players follows the debt limit strategy, and then verify the IC and IR conditions. In verifying the IC conditions, we also show that all transactions that occur are efficient. Lastly we confirm that (5) is always met by some distributions of costs.

1. Computation of expected future value and soft prices

Suppose the debt limit has never been exceeded. Given that all transactions go through if the debt after transaction falls within $[-L, L]$ and the seller is able to help, then for $-L \leq k \leq L$,

$$\begin{aligned} V_k = \quad & \frac{1}{2}\left[\frac{\pi}{B}\sum_{b=1}^{L+k}(b+\delta V_{k-b}) + \left(1-\frac{\pi}{B}(L+k)\right)\delta V_k\right] \\ + \quad & \frac{1}{2}\left[\frac{\pi}{B}\sum_{b=1}^{L-k}(-\widehat{c}_b+\delta V_{k+b}) + \left(1-\frac{\pi}{B}(L-k)\right)\delta V_k\right] \end{aligned}$$

Compared to the V_k in (6) of the proof of Proposition 2, the expected cost αb is replaced by $\widehat{c}_b$.

Following similar steps as in Proposition 1, we get:

$$p_{k+1,k} = \frac{L+k+1+\widehat{c}_{L-k}}{2B\left(\frac{1/\delta-1}{\pi}\right)+2L+1} \tag{11}$$

Again, since $p_{k+b,k} = \sum_{r=0}^{b-1} p_{k+r+1,k+r}$ for $b = 1, .., L-k$,

$$p_{k+b,k} = \frac{b\left(L+k+\frac{b+1}{2}\right)+\sum_{r=0}^{b-1}\widehat{c}_{L-k-r}}{2B\left(\frac{1/\delta-1}{\pi}\right)+2L+1} \tag{12}$$

2. Verification of IC and IR

Now we show that under (5) the players indeed find it profitable to trade whenever the debt level will remain within L. For $k = 0, \pm 1, .., \pm L$, the IC and IR conditions are restated as follows:

IC (buyer): $p_{k,k-b} < b$ if $b = 1, .., L+k$

$$\text{IC (seller): } p_{k+b,k} > \overline{c}_b \text{ if } b = 1,..,L-k$$

$$\text{IR: } V_k > 0$$

Note that the right-hand side of (5) in the proposition is just $p_{k+b,k}$. Therefore (5) is equivalent to IC (seller).

We now show that IC (buyer) is guaranteed by IC (seller). First note again that IC (buyer) is equivalent to:

$$p_{k+1,k} < 1 \ , \ k = -L, -L+1, .., L-1$$

Use (11) and rearrange terms, IC (buyer) becomes:

$$\widehat{c}_{L-k} < 2B\left(\frac{1/\delta - 1}{\pi}\right) + L - k$$

Substitute b for $L-k$, IC (buyer) can be rewritten as:

$$\widehat{c}_b < 2B\left(\frac{1/\delta - 1}{\pi}\right) + b \ , b = 1, .., 2L$$

Therefore showing that all transactions that occur are efficient (i.e. $\overline{c}_b < b$) is more than sufficient to prove that IC (buyer) holds.

We will prove the efficiency by induction. From (5) and (11), pick $k = L-b$ and $b = 1$,

$$\overline{c}_1 < p_{L,L-1} = \frac{2L + \widehat{c}_1}{2B\left(\frac{1/\delta-1}{\pi}\right) + 2L + 1} \implies \overline{c}_1 < \frac{2L}{2B\left(\frac{1/\delta-1}{\pi}\right) + 2L} < 1$$

So all transactions of single-unit benefits are expected to be efficient.

Next, assume efficiency ($\overline{c}_r < r$) holds for $r = 1, .., b-1$, where $b \leq 2L$, then from IC (seller) and (12),

$$\overline{c}_b < p_{L,L-b} \quad = \quad \frac{b\left(2L + \frac{1-b}{2}\right) + \sum_{r=1}^{b} \widehat{c}_r}{2B\left(\frac{1/\delta-1}{\pi}\right) + 2L + 1}$$

But

$$\sum_{r=1}^{b} \widehat{c}_r \leq \sum_{r=1}^{b} \overline{c}_r < \sum_{r=1}^{b-1} b + \overline{c}_b = \frac{b\,(b-1)}{2} + \overline{c}_b$$

(The second step holds as a result of the above assumption.) Combine the two inequalities and rearrange terms,

$$\overline{c}_b < \frac{2Lb}{2B\left(\frac{1/\delta-1}{\pi}\right)+2L} < b$$

Therefore efficiency holds for transactions involving benefit b if it holds for transactions of all lower benefits. By induction, all transactions that occur are efficient. IC (buyer) is satisfied.

IR can be verified in similar manner as in Proposition 2.

Applying the one-shot deviation principle as in the proof of Proposition 2, the debt limit equilibrium is shown to exist.

3. Existence of distribution of costs that support the equilibrium

To show that (5) can always be satisfied by some set of $\{\overline{c}_b, \widehat{c}_b,\ b = 1, .., 2L\}$ so that the equilibrium exists, consider the case where $\overline{c}_b = \overline{c}$ and $\widehat{c}_b = \overline{c}/2$ for all b (for example, $\widetilde{c}_b$ is uniformly distributed over $(0, \overline{c})$ for all b). Since $p_{k+b,k}$ is increasing in b, if (5) is satisfied for $b = 1$, then it is for all b. This condition simply requires

$$\overline{c} < \frac{L+k+1+\overline{c}/2}{2B\left(\frac{1/\delta-1}{\pi}\right)+2L+1} \Leftrightarrow \overline{c} < \frac{L+k+1}{2B\left(\frac{1/\delta-1}{\pi}\right)+2L+\frac{1}{2}}$$

Since $k \geq -L$, picking a $\overline{c}$ smaller than $\frac{1}{2B\left(\frac{1/\delta-1}{\pi}\right)+2L+\frac{1}{2}}$ guarantees (5) is met. □