OCCUPATIONAL DIVERSITY AND ENDOGENOUS INEQUALITY

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This paper constructs a model of intergenerational transmission of inequality via financial bequests and human capital investments in the presence of borrowing constraints. It extends existing models in various directions: investment returns are endogenously determined, there are no indivisibilities in investment, and financial and human capital co-exist. A unique steady state is shown to exist. The steady state involves wealth inequality across families (and endogenous nonconvexity in returns to human capital) if and only if the range of training costs across occupations is large, relative to technology and preference parameters.

Keywords: persistent inequality, intergenerational bequests, convergence.

1. Introduction

Two fundamental questions in the theory of income distribution have recently received much attention.

[I] Do markets possess some intrinsic tendency to equalize or disequalize the fortunes of different households or societies?

[II] Does history matter? That is, are the same economic fundamentals consistent with several steady states which vary in overall inequality and aggregate output?

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[I] lies at the heart of an entire literature on convergence. [II] forms the basis of an important argument, commonly employed in development economics, that historical "accident" can cast long shadows on subsequent paths of development.

To pose these questions precisely, one needs to clarify the term "markets", as well as what it means to "equalize" or "disequalize". By "markets" we refer to price-taking market-clearing economies, in which wealth is transmitted across generations via financial bequests and educational investments. In keeping with a long tradition that we discuss below, we presume that future generations cannot be held responsible for the debts of their ancestors, so that all bequests must be nonnegative.[2]

As for what it means to "equalize" or "disequalize", we believe that the context of such a definition must be one in which the dynamics of income distribution are generated *only* by the operation of the market, in the (hypothetical) absence of any heterogeneity or stochastic shocks in abilities, tastes or opportunities. This allows a clear conceptual separation between the role of the market *per se* and heterogeneity or random events in generating or propagating inequality.

For dynastic households, this allows us to frame question I as follows: consider two households with similar abilities and tastes but different wealth. Does the market by itself, *in the absence of any stochastic shocks*, induce the wealths of their descendants to draw ever closer together, and their difference to eventually vanish? If so, say that the market is *equalizing*. Conversely, the market is *disequalizing* if — in the absence of heterogeneity or random events — it causes the wealths of equal or near-equal households to separate over subsequent generations, and such wealth differentials persist in the long run. In particular, long run equality would *never* be a possible outcome if the market is disequalizing.

Or perhaps the market simply turns a blind, neutral eye to wealth differences. It maintains wealth differences when there are any, and conversely does nothing to disturb a

[2]Loury (1981) summarizes it clearly: "Legally, poor parents will not be able to constrain their children to honor debts incurred on their behalf."

relatively equal state of affairs. Such neutrality, at least over a range, might imply an affirmative answer to question II.

Papers such as Solow (1956), Brock and Mirman (1972), Becker and Tomes (1979, 1986), Loury (1981) — as well as a large empirical literature — emphasize equalization. Equilibrium inequality, if any, is just the outcome of a tussle between the tendency to converge and ongoing shocks to abilities or opportunities ("luck", generally speaking).

Papers such as Ray (1990), Ljungqvist (1993), Freeman (1996), Bandyopadhyay (1997), Mookherjee and Ray (2003) emphasize disequalization, arguing that markets *must* separate individual fortunes: *all* steady states involve interhousehold inequality that persists across generations. These models, well as the important contributions of Matsuyama (2000, 2004) that we discuss in more detail below, rely on "symmetry-breaking". In particular, the demand by the market for a diversity of occupations forces individuals in identical or near-identical situations to make distinct choices of profession, with implications for the subsequent emergence of inequality.

In between these two approaches lies a third literature (Banerjee and Newman (1993), Galor and Zeira (1993), Ray and Streufert (1993), Aghion and Bolton (1997), Piketty (1997), Lloyd-Ellis and Bernhardt (2000), and Ghatak and Jiang (2002)).[3] These "neutral" models permit both initial inequalities and initial equalities (and perhaps various degrees of inequality) to persist, both between households and — or, in a familiar interpretation — across countries. Typically, history determines where a household or society ends up in the long run.[4]

[3]Aghion and Bolton's model is characterized by an ergodic wealth distribution, owing partly to some strong parameter restrictions they impose. In the absence of such restrictions, their model would also belong to the general class of "neutral" models.

[4]There are also models in which the steady state distribution of wealth is fully indeterminate — see, e.g., Chatterjee (1994). Here the investment frontier that each household faces is linear (rather than strictly convex as in Loury (1981)), so with a dynastic bequest motive each family wants to maintain its wealth indefinitely, and a steady state is compatible with *any* interfamily wealth distribution. This arbitrariness disappears if the bequest motive is modified, as in Becker and Tomes (1979), to a paternalistic motive where parents care about the wealth of only their next generation, rather than the infinite succession of subsequent generations. Steady state is then consistent only with perfect interfamily wealth

Equalization models rely on the *assumption* that the marginal return to human capital is decreasing. When different occupations are imperfect substitutes for one another such an assumption is untenable, because it fixes relative prices *ex ante*. On the other hand, theories of disequalization rely on the absence of financial bequests. The neutrality models assume that there are indivisibilities in investments. We study a framework which simultaneously eliminates these special features. In this model, different occupations enter as imperfect substitutes into production, perfectly divisible financial bequests are available, and the occupational structure is "rich", in the sense that for every occupation, there exists another which requires similar levels of human investment to acquire. We suppose, moreover, that a dense subset of these occupations is essential in production. (See Conditions [R] and [E] in Section 3 below.)

We reexamine questions I and II in this framework. But the objective is not just greater generality for its own sake: the exercise yields new insights into the role of the market in generating or ameliorating inequality in the long run.

The main object of our inquiry is a *steady state*, in which the societal distribution of occupational choices, the wealth distribution, all factor prices, and aggregate output are unchanging over time.

A preliminary result of central importance to what follows is Proposition 1. This proposition provides a complete characterization of steady states in our model, whether or not they involve inequality. The proposition establishes that a steady state must be associated[5] with a wage function that has a *two-phase property*: over a stretch of occupational training costs it is linear, after which it turns upward with higher marginal rate of return. The proposition provides a precise description of the boundary that separates the two phases(either phase may be empty), and a full description of the nonlinear second phase.

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equality (under the assumptions made by Becker and Tomes on the strength of parental altruism). Alternatively, if investments are subject to diminishing returns at the level of each household, then again perfect long run equality is predicted, irrespective of the bequest motive. The discussion hereafter focuses on versions with a determinate theory of distribution.

[5]The word "associated" involves some technical detail which is clarified in part (a) of Proposition 1.

This characterization allows us to provide necessary and sufficient conditions for the existence of a steady state with positive aggregate output (Proposition 2).

Now turn to the two questions that motivate this paper. First, if the occupational structure is rich, *there is a unique steady state* (Proposition 3). This steady state may or may not involve persistent inequality but in either case *history cannot matter in determining long-run outcomes for the economy as a whole*, at least via the channel studied in this class of models.[6]

This observation is particularly striking because the (assumed) presence of a rich set of occupations does *not* guarantee that the set of bequest-return opportunities available to a parent is convex, nor indeed that it has any particular shape. Richness assures us that a family can fine-tune its human capital investments, but the *payoffs* from those investments will ultimately be determined by factor prices for each occupation, and these are endogenous. In principle, then, a household's investment frontier may well exhibit diminishing returns(as in Brock and Mirman (1972), Loury (1981) or Becker and Tomes (1986)), or it may display significant nonconvexities (as in Majumdar and Mitra (1982)) or even an approximation of indivisibilities (Banerjee and Newman (1993), Galor and Zeira (1993)). There is no pre-judgment about whether convexities are the rule rather than the exception.[7]

Yet, despite the ability of the model to generate nonconvexities, the economy as a whole cannot be locked into several different steady states. In the discussion surrounding Proposition 3, we provide an intuitive basis for this result, and also relate it to the more

[6]Our conclusion that a unique steady state implies the long-run absence of historical effects involves a leap of faith that needs to be addressed in future research. We are presuming that steady states capture all there is to know about the long-run behavior of our model economy. This is only true if dynamic equilibria indeed converge to such steady states. Ray (2006) and the working-paper version of this article (Mookherjee and Ray (2006)) provide some support for this conjecture, but a full investigation of the dynamics in the case of a rich occupational structure is yet to be carried out.

[7]Contrast this approach — as we will in Section 2 — to one in which human capital can be reduced to efficiency units. Then the relative prices across occupations are pinned down *ex ante*, and this flexibility is lost.

limited uniqueness theorem in Mookherjee and Ray (2003), in which financial bequests are not permitted, and dynastic preferences are assumed.

As the discussion following this proposition makes clear, however, the long-run fate of an *individual* family is fundamentally path-dependent. Our model certainly does not rule out history-dependence at this level. It is the overall distribution of occupations, wealth, and output that Proposition 3 pins down.

We then turn our attention to question I. Will the steady state — unique by Proposition 3 — involve equality or inequality? Proposition 4 applies the characterization of Proposition 1 to obtain a principal result of the paper. We provide a necessary and sufficient condition on the primitives of the model for the steady state to involve inequality. This condition, which we call the *widespan condition*, may be stated in the following, imprecise way: The "span" of the occupational structure (as measured by the range of training costs) is large relative to the "strength" of parental bequests.

This is a rough statement, because the "strength" of parental bequests is actually a composite of various parameters: the intensity of the bequest motive and the interest rate among them, and in an indirect way that will become very clear below, even the technology of production. Moreover, the odd juxtaposition of "occupational span" and "parental bequests" seems a bit like comparing objects in two different units; this too will become clear in the main text.

Suitable applications of the wide-span condition yield a variety of implications. A particularly interesting one is that countries poorer than others owing to (exogenously) lower economy-wide TFP are more prone to disequalization. At the same time, the condition also suggests — somewhat in contrast to the previous observation — that faster TFP growth makes an economy also more vulnerable to disequalization. The wide-span condition also allows us to study the inequality effects of technological change.

Of additional interest are results concerning the shape of earnings functions in steady state. If occupational span is narrow, the unique steady state must involve equality of returns to human and physical capital: in this case households face a linear investment frontier and the equalization hypothesis is correct. If the span is wide, however, the "equilibrium investment frontier" is *necessarily* non-convex: the rate of return to human capital equals the returns to physical capital over low ranges of human capital, and it strictly exceeds the return to physical capital over higher levels of human capital.

Indeed, for a broad class of utility functions (including those with constant elasticity), the rate of return to human capital is monotonically rising over the high range, with the highest returns arising at the top end of the occupational distribution. Higher interest rates are associated with lower earnings in the most unskilled occupations, and higher marginal rates of return to human capital at all levels. These predictions contrast sharply with those of Becker and Tomes (1986), where the (technologically determined) returns to human capital are initially higher than the return on physical capital (and independent of the interest rate), and fall monotonically with the level of human capital. Our prediction regarding financial bequests are also different, though this requires a more nuanced discussion that we postpone for now.

In closing this introduction, we wish to relate our paper to the work of Matsuyama (2000, 2004), which is also concerned with questions of symmetry-breaking, equalization and disequalization. Following Piketty (1997), Matsuyama studies synmmetry-breaking in capital markets when there are explicit borrowing constraints. For instance, Matsuyama (2000) shows that under some parametric restrictions, an equal steady state is not possible. Some poor agents must endogenously emerge as lenders, other rich agents as investors. Similarly, Matsuyama (2004) shows that even if all agents converge to the same steady state (for the usual reasons of assumed convexity), the introduction of (imperfect) financial markets across those agents may lead them to separate from one another.

While Matsuyama's concerns are broadly similar, they are in an entirely different arena. His symmetry-breaking comes from the need to place agents on opposite sides of the capital market. In our language, there are just two "occupations" in Matsuyama: investors and lenders. This is entiely different from what we do, even if we adopt the interpretation that occupations are different firms (see Section 9). There, (our kind of) symmetry-breaking would arise simply because all sorts of different firms need to be active.

Our research may therefore be viewed as complementary with that of Matsuyama's. The models studied are different, and our other central results (uniqueness with a continuum of occupations, or the equilibrium shape of wage functions), have no counterpart at all in Matsuyama's work. (Likewise, Matsuyama studies issues, such as enforcement constraints in borrowing, which have no counterpart in our research.)

The paper is organized as follows. Section 2 contains some preliminary observations. Section 3 describes the model; Section 5 presents the characterization result, and Section 6 goes on to study existence and uniqueness of steady states. Section 7 takes up the question of disequalization versus equalization. Section 8 discusses implications and applications of these results. Section 9 then discusses various extensions of the model: indivisibilities in occupational investments, a closed economy where interest rates are endogenously determined, possible non-essentiality of some occupations, existence of credit rationing and contexts where different occupations correspond to different intermediate goods produced by firms of differing size and setup costs. Section 10 concludes, and the Appendix collects technical proofs.

2. Occupational Structure and Disequalization

Economists have traditionally employed a simple shorthand for the study of occupational diversity, which is to reduce different qualifications and skills to aggregate quantities of "human capital". In other words, all human capital is — *even before we write down the definition of equilibrium for the society in question* — commonly expressible in some common efficiency unit.[8] This approach is summarized by Becker and Tomes in their 1986 paper:

[8] So, for instance the investment choice in Loury's model is interpreted as a choice of "how much" education to acquire: there is no formal difference between human and physical capital. The so-called endogenous growth models (see, e.g., Lucas (1988)) continue, by and large, to retain this shorthand.

> "Although human capital takes many forms, including skills and abilities, personality, appearance, reputation and appropriate credentials, we further *simplify* by assuming that it is homogeneous and the same "stuff" in different families." (Becker-Tomes (1986, p.56), emphasis ours)

The crucial assumption is that the relative returns to different occupations are exogenous, so that the reduction to efficiency units can be carried out separately from the behavioral decisions made in the population. The implications of such an assumption can be quite drastic.

The following example is based on Ljungqvist (1993), Freeman (1996) and Mookherjee and Ray (2003). Suppose that aggregate production is a CRS function of just two inputs: skilled and unskilled tasks, satisfying Inada conditions in each. There are two occupations: skilled and unskilled labor. The latter can only do the unskilled tasks. Wages in each occupation equal their respective marginal products, and so depend on relative supplies of workers in the two occupations. Skill acquisition requires a fixed parental investment in education. Assume that this is the only way a parent can transfer wealth to their children, i.e., there are no financial bequests.

Now observe that even if all parents in the economy have identical wealth and preferences, *they cannot all leave the same bequests*. The reason is simple. If every parent keeps their child unskilled, there will be no skilled people in the next generation, raising the return to skilled labor enough that investment in skill will be the optimal response. Conversely, if all children are skilled, the return to skill will vanish, killing off the investment motive.[9] Hence even if all families start equal in generation 0, some will invest and others will not; in the next generation their fortunes must separate.

Just who goes in one direction and who in another is entirely accidental, but such accidents will cast long shadows on dynastic welfare. Indeed, the two directions are

[9]If skilled labor cannot perform unskilled tasks, the unskilled wage will become very high by the Inada conditions. But even if they can perform unskilled tasks, this will equalize the two wage rates. Either interpretation has the same outcome.

utility-equivalent for generation 0, but not for generation 1! Furthermore, in succeeding generations wealthier parents will have a greater incentive to train their children, so that the "primitive inequality" that sets in at the first generation will be reinforced: children of skilled parents will be more likely to acquire skills themselves. The logic of "symmetry-breaking" implies that every steady state in this example must involve persistent inequality. The endogeneity of occupational returns is central to this argument.[10]

In sharp contrast, observe that the same argument does not apply to activities in which each unit is a perfect substitute for another. For instance, if shares in physical capital can be divisibly held, everyone can derive the very same rate of return on each unit. But a single individual cannot hold an arbitrarily fine portfolio of different occupations.[11]

With perfectly substitutable units, the rate of return must be arbitraged to linearity, and the Becker-Tomes logic does apply: such bequests will tend to equalize the fortunes of successive generations. Therefore the *joint* presence of imperfect-substitute investments, such as occupations, and perfect-substitute investments, such as money, provokes the following question: does the logic of symmetry-breaking still apply? Or can the presence of "financial bequests" nullify the symmetry-breaking in occupations or other inlienable activities? This, among others, is the question we pose and answer below. In arriving there, moreover, we obtain a striking and empirically testable theory of steady-state wage functions.

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[10] See e.g., Katz and Murphy (1992) for the responsiveness of US skill premia to relative supply of skilled workers.

[11] This discussion suggests, then, that the correct dividing line is not between "physical" and "human" bequests, but rather bequests that result in endowments that are alienable (e.g. money) and endowments that are not (e.g. occupations). The latter may include transfers of physical assets such as a family business which is not incorporated — perhaps for reasons of moral hazard or simply the lack of development of a stock market. These transfers are no different from human bequests in their implications for disequalization, and should be included in the category of "occupational bequests". Section 9 discusses this in more detail. The remaining, alienable component we shall refer to as "financial bequests".

3. Model

3.1. **Occupations and Training.** There is a compact Borel measurable space $\mathcal{H}$ of *occupations* that will be used in the production of a single, aggregative final good. There is an exogenous *training cost* $x(h)$ for occupation $h \in \mathcal{H}$, denominated in units of final output.[12] The following richness assumption will hold throughout the paper:

[R] The set of all possible training costs is a compact interval of the form $[0, X]$.

We justify [R] by noting that while there are large differences in training costs between unskilled occupations (such as farm workers or manual jobs) and skilled occupations (such as engineers, doctors and lawyers), there are also many semi-skilled occupations (technicians, nurses and clerks) with intermediate training costs and wages. Besides, there are large differences in the quality of education within any given occupation, which translate into corresponding differences in education costs and wages.

The methodological innovation in an assumption such as [R] is that it allows families to fully fine-tune their investments. Whether or not their investment set is convex depends, then, not on assumed indivisibilities in training costs but in the endogenously determined factor price schedule.

Note that [R] includes a bit more than occupational richness. It states that there is an occupation with zero training cost, a restriction that we impose only for expositional ease.

3.2. **Production.** A single aggregate output is produced by physical capital[13] and individuals who hold occupations in $\mathcal{H}$.

[12] As in Mookherjee and Ray (2003), this may be generalized to allow training costs to depend on the pattern of wages. We conjecture that the principal qualitative results of this paper will continue to hold in that setup.

[13] As long as capital goods are alienable and shares in them can be divisibly held, having several capital goods makes no difference to the analysis.

Output y (net of the undepreciated capital stock) is produced by a continuous,[14] strictly quasiconcave CRS production function $y = f(k, \boldsymbol{\lambda})$, where k is physical capital and $\boldsymbol{\lambda}$ is an occupational distribution (a finite measure on $\mathcal{H}$). It will be helpful to interpret different occupations as corresponding to different kinds of human capital; in Section 9 we explain how to extend this interpretation to the ownership of closely-held firms of differing scales that produce different intermediate goods.

We will assume that "every occupation" is essential in production:

[E] For every subset $C \subseteq [0, X]$ of positive Lebesgue measure, if the occupational distribution has zero value over every occupation h with $x(h) \in C$, then no output can be produced.

So we don't ask that every occupation be essential, only that (almost) every training cost in $[0, X]$ has an essential occupation attached to it. Observe that conditions [R] and [E] really go together as a pair: without some restriction like [E], [R] can always be trivially met by simply inventing useless occupations to fill up the gaps in training costs.

Together, [R] and [E] imply that whenever positive output is produced, the inhabited range of "equilibrium training costs" is always equal to $[0, X]$.

3.3. **Prices and Firms.** Firms maximize profits at given prices. Normalize the price of final output to 1. Let $\mathbf{w} \equiv \{w(h)\}$ denote the wage function, and $\mathbf{p} \equiv (r, \mathbf{w})$ the factor price function, where r is the rate of interest. Denote by $c(\mathbf{p})$ the unit cost function.

By constant returns to scale, profit maximization at positive output is possible if and only if $c(\mathbf{p}) = 1$; in that case call $\mathbf{p}$ a *supporting price*. If $(k, \boldsymbol{\lambda})$ is a profit-maximizing choice under the supporting price $\mathbf{p}$, we will refer to it as an *associated* input vector.

In this paper, we assume that the rate of interest r is exogenously given and time-stationary. One simple interpretation is that capital is internationally mobile and that our economy is a price taker on the world market. Under this interpretation we also

[14]Endow the space of all nonnegative finite measures on $\mathcal{H}$ with the topology of weak convergence. We ask that output be continuous with respect to the product of this topology and the usual topology on k.

assume, in effect, that *people* are not internationally mobile: the wage function $\mathbf{w}$ will be determined domestically. The assumption of capital mobility is largely for expositional simplicity; see Section 9.2 and Mookherjee and Ray (2006) for extensions.

Now that r may be treated as a parameter, say that $\mathbf{w}$ is a *supporting wage* if $\mathbf{p} = (r, \mathbf{w})$ is a supporting price.

3.4. **Families.** There is a continuum of families indexed by $i \in [0, 1]$. All families are *ex ante* identical, so we endow $[0, 1]$ with Lebesgue measure ν. Each family i has a single representative at each date or generation, indexed by t. Call this agent (t, i).

To describe wealth dynamics, consider a member of generation t. She begins adult life with a financial bequest b and an occupation h, both "selected" by her parent. The latter is obviously shorthand for the assumption that the parent bears the costs of upbringing and education (the *child* can select the particular occupation with no difference to the formal analysis). The overall wealth of our generation-t adult is then $W \equiv b(1+r)+w_t(h)$, where $w_t(h)$ is the going wage for occupation h at date t.

The agent correctly anticipates factor prices $\mathbf{p}_{t+1} \equiv (r, \mathbf{w}_{t+1})$ for the next generation $t+1$, and selects her own financial and educational bequests (b', h') to maximize

$$U\left(W - x(h) - b'\right) + V\left((1+r)b' + w_{t+1}(h')\right) \tag{1}$$

subject to the no-intergenerational-debt constraint $b' \geq 0$. We assume that U and V are smooth, increasing and strictly concave, and that U has unbounded steepness at 0.

Now b' and h' become the financial and educational inheritance of her child — generation $t+1$ — and the entire process repeats itself *ad infinitum*.

In (1), U is obviously defined on lifetime consumption (wealth minus bequests), while V is defined on the child's lifetime wealth. So the bequest motive implicit in (1) is more sophisticated than "warm-glow" bequests — assumed in much of the literature — in which parents care about the bequest *per se* rather than what it does for the future wealth of their children.[15] Bequests will therefore be endogenously determined as a function of anticipated factor prices.

However, the bequest motive is not "dynastic": parents do not internalize the full consequences of their bequest on the utility of their children. In other words, V is *not* a value function. For the exercise we wish to conduct here, nonpaternalistic dynamic preferences have a serious drawback: they are inconsistent with the "limited persistence" property imposed by Becker and Tomes; a property we wish to incorporate here. See Section 4 for more discussion.

The condition $b' \geq 0$ is a fundamental restriction stating that children cannot be held responsible for debts incurred by their parents. The capital market is active in all other senses: households can make financial bequests at the going rate r, and firms can freely hire in capital at the very same rate.[16]

3.5. **Equilibrium.** Begin with an initial distribution of financial wealth and occupational choices. A *competitive equilibrium* given these initial conditions is a sequence of wage functions $\mathbf{w}_t, t = 0, 1, 2, \ldots$ and occupational distributions $\boldsymbol{\lambda}_t$, $t = 0, 1, 2, \ldots$, as well as occupational and bequest choices for each generation in each family — $\{h_t(i), b_t(i)\}$ — such that for each t and each family i:

(a) person (t, i) chooses $(b_{t+1}(i), h_{t+1}(i))$ to maximize the utility function in (1), given that her own starting wealth equals $(1 + r)b_t(i) + w_t(h_t(i))$;

(b) these decisions aggregate to $\boldsymbol{\lambda}_t$ at each t:

$$\lambda_t(H) = \nu\{i \in [0, 1] | h_t(i) \in H\}$$

for every Borel subset H of occupations, and

(c) $\mathbf{w}_t$ is a supporting wage, with associated input vector $(\boldsymbol{\lambda}_t, k_t)$ for some choice of k_t.

[15]Most models with warm-glow bequests also assume Cobb-Douglas preferences between own-consumption and the size of the bequest, implying that the total bequest size is a constant fraction of parental wealth, independent of prices.

[16]Section 9 explains how the model can be extended to the context where households can also borrow, subject to credit limits.

Observe that equilibrium conditions place no restrictions on k_t. Because there is international capital mobility, financial holdings by households need bear no relation to capital used in production.[17]

A *steady state* is a competitive equilibrium with stationary prices and distributions; $(\mathbf{w}_t, \boldsymbol{\lambda}_t) = (\mathbf{w}, \boldsymbol{\lambda})$ for all t, and strictly positive output.[18]

Our paper exclusively studies steady states. Though we conjecture that every competitive equilibrium must converge to some steady state, this topic must be the subject of future research.

The following observation is useful: in steady state, the total wealth of *every* family, not just of the economy as a whole, must be stationary.[19] Its proof, based on the familiar single-crossing argument that wealthy parents are more willing to invest in their children's wealth, is relegated to the Appendix.

OBSERVATION **1.** *The wealth of every family is stationary in any steady state.*

By Observation 1, not just aggregate wealth but the *distribution* of cross-family wealth is well-defined in each steady state.[20] In what follows, an *equal steady state* will refer to a steady state with a degenerate wealth distribution; all other steady states will be called *unequal.*

4. A BENCHMARK

A special case of this model is the following elementary textbook exercise: *only* financial bequests are possible (earning interest r), and everyone earns a fixed wage w.

[17]When there is no international capital mobility, k_t must equal the aggregate of financial holdings, and r must adjust to assure this equalization in equilibrium.

[18]A steady state with zero output is also a possibility but we will impose conditions that rule this out. See Condition [P] in Section 6 and the accompanying discussion in footnote 28.

[19]In general, such an assertion is not true of the financial wealth or the occupational choice of a family, which may vary over time.

[20]Some or all of these wealths may be infinite, in principle.

Stochastic shocks apart, this is exactly the specification for the Becker-Tomes (1979) model, which assumes a linear rate of return to parental investment in children.[21] We will henceforth refer to this special case as the *Becker-Tomes benchmark*.

In this special case, a parent with wealth W simply selects $b \geq 0$ so as to maximize

$$U\left(W-b\right)+V\left(w+(1+r)b\right).$$

Let the resulting wealth of the child be denoted $\tilde{W} \equiv w+(1+r)b$. We may write $\tilde{W}$ as a function of W, w and r: $\tilde{W}(W; w, r)$. By our assumptions, $\tilde{W}$ is fully characterized by the first order conditions

$$U'\left(W-\frac{\tilde{W}-w}{1+r}\right) \geq (1+r)V'(\tilde{W}), \tag{2}$$

with equality if $\tilde{W} > w$.

It is obvious that $\tilde{W}(W; w, r)$ is nondecreasing and continuous in W. So an iteration of this mapping from any initial condition $W > 0$ will yield long-run wealth starting from W. Call this long-run wealth Ω. Passing to the limit in (2), it is trivial to see that if $w \leq \Omega < \infty$,

$$U'\left(\frac{r\Omega+w}{1+r}\right) \geq (1+r)V'(\Omega), \text{ with equality if } \Omega > w. \tag{3}$$

Becker and Tomes impose the following restriction on bequest behavior: $\frac{\partial\tilde{W}}{\partial W} \in (0,1)$, justifying it by available empirical evidence. This implies that the wealth of all families will converge to a common limit Ω, *independent of initial wealth.* To ensure that our larger model is consistent with this key equalization property of financial bequests, we impose a similar (though weaker) restriction in this special case:

[21]A more sophisticated version of their model is Becker and Tomes (1986), in which bequests in the form of human capital are also permitted, but human capital is *a priori* reduced to efficiency units and it is *assumed* that the rate of return to successive units of human capital is declining. In that variant, all families will aim to invest the (same) amount of human capital before turning to linear financial bequests. Indeed, we do not claim that our elementary textbook exercise captures the full implications of the Becker-Tomes models. An important theme in these papers is the interplay between luck and convergence, an issue that is not of relevance here.

[LP] *Limited Persistence.* For any $r > -1$ and $w \geq 0$, there is at most one solution in $\Omega \geq w$ to (3).

As in Becker-Tomes, [LP] also implies that limit wealth $\Omega(w, r)$ is well defined and independent of starting wealth as long as as that starting wealth strictly exceeds w (that is, as long as starting financial wealth is positive). Given [LP], here is how we define that limit wealth: Set $\Omega(w, r)$ equal to Ω, where Ω solves (3), in *all* cases except the one in which

$$U'\left(\frac{r\Omega + w}{1+r}\right) > (1+r)V'(\Omega)$$

for all $\Omega > w$, in which case set $\Omega(w, r) = \infty$.[22] The important point is that $\Omega(w, r)$ is the same no matter what the (positive) level of inital wealth is.

This condition is less restrictive than the original Becker-Tomes assumption on $\frac{\partial W'}{\partial W}$, though the reader is welcome to keep the stronger restriction in mind (as long as it is understood that an implicit restriction on the interest rate is also implied thereby).

Condition [LP] is extremely easy to check. The following observation illustrates this by studying the HARA class of preferences.

OBSERVATION **2.** *Suppose preferences satisfy the following restriction: There exists $\delta > 0$ such that $V = \delta U$, and U belongs to the* HARA *family:*

(H) $$-U''(c)/U'(c) = 1/(\alpha + \beta c),$$

where $(\alpha, \beta) \geq 0$ and nonzero. Then the limited persistence property is satisfied, with the single exception in which alpha $= 0$ and $w = 0$, in which case it is satisfied for all but one value of r.

Provided that preferences satisfy a constant discount rate property, Observation 2 states that [LP] holds for utility functions that are iso-elastic or exponential, or belong to the HARA class which nests these as special cases.

[22] The only subtlety here is one in which (3) holds with equality at $\Omega = w$, while the inequality $>$ holds for all $\Omega > w$. In this case both limits w and ∞ are potential candidates, but the correct limit for starting $W > w$ is easily seen to be the latter. This is a nongeneric case of little import but in any case our definition handles it.

It helps to illustrate bequest behavior in the setting with financial bequests alone for the special case of iso-elastic utility with discounting: $U(c) = (c^{1-\sigma} - 1)/(1-\sigma)$ with $\sigma > 0$, and $V \equiv \delta U$, with $\delta \in (0, 1)$. Define $\rho \equiv [\delta(1 + r)]^{1/\sigma}$. Intergenerational wealth movements in the Becker-Tomes benchmark with stationary (w, r) then takes the form:

$$\tilde{W} = \frac{(1+r)\rho}{1+\rho+r}W + \frac{\rho}{1+\rho+r}w$$

if $W \geq \frac{w}{\rho}$, and $\tilde{W} = w$ otherwise. This allows us to calculate limit wealth:

$$\Omega(w,r) = \begin{cases} w & \text{if } \rho \leq 1, \\ \frac{\rho}{1-r(\rho-1)}w & \text{if } \rho \in (1, 1+\frac{1}{r}), \\ \infty & \text{if } \rho \geq 1 + \frac{1}{r}. \end{cases} \tag{4}$$

If $\rho \leq 1$, there are no (limiting) financial bequests in steady state in the Becker-Tomes benchmark, and $\Omega(r, w) = w$. Limit wealth $\Omega(w, r)$ is finite only if $\rho \in (1, 1 + \frac{1}{r})$.

If r is high enough that this condition is not satisfied, limit wealth is infinite: our version of the limited persistence property is satisfied,[23] whereas the Becker-Tomes version is not. We only point this out to emphasize that the Becker-Tomes version of limited persistence imposes more than a restriction on preferences, but in any case we are also fundamentally interested in the case in which limit wealth is finite.[24]

We end this section by pointing out that dynastic preferences are inconsistent with the limited persistence assumption. By the envelope theorem, the derivative of the value function is just the marginal utility of (equilibrium) consumption. At a steady state, then, these marginal utilities drop out from the Euler equation, and the conditions become independent of wealth. In particular, an increase in parental steady state wealth translates into an equal increase in child wealth, so that [LP] fails. Dynastic preferences do not allow us to take the Becker-Tomes postulates fully on board.

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[23]Notice that there are two limit wealths in this case, one at infinity and one at zero. But there is only one limit wealth provided we start with strictly positive wealth, and this is what [LP] requires.

[24]While the case of unbounded limiting wealth is formally a special case, ongoing growth really calls for a different model, ideally one in which training costs are endogenous as well.

Why do we want to incorporate [LP] in the first place? One answer is that it is empirically attractive. That may be so, but we do not impose [LP] for this reason. We do so because we want to show that this assumption generates a form of the steady state wage function that can never exhibit diminishing returns! If at all that wage function is nonlinear, it must be *non*convex at least over a region. This shows that the *assumption* that returns to human capital must be diminishing (Becker and Tomes (1986)) merits careful scrutiny, to say the least. None of these points emerge with dynastic preferences, and this — apart from financial bequests — is another basic difference from the analysis in Mookherjee and Ray (2003).

5. A Characterization of Steady States

We begin the main analysis by describing important properties of equal and unequal steady states.

Fix a steady state. Say that an occupation (or training cost) is *inhabited* if some family chooses that occupation (or incurs that training cost). By conditions [R] and [E], we know that every steady state (which has positive output, by definition) must exhibit a full measure of inhabited training costs. Suppose that we can alter the wage function on the small set of uninhabited occupations without changing *any* of the observed features of the steady state. Then we will say that the new steady state wage function (which only differs by specifying different wages for uninhabited occupations) is an *equivalent representation* of the old.

One particular representation is of interest, in which all occupations with the same training cost command the same wage. As Lemma 3 in the Appendix formally records, this *must* be true of any two (equally costly) occupations which are inhabited. With some abuse of notation, we shall go back and forth between $w(h)$ and the representation $w(x)$.

Part (a) of Proposition 1 below asserts that for every steady state there is an equivalent representation with a continuous wage function $w(x)$, and from this point on we will refer without qualification to that equivalent representation.

These are just technical matters. More substantively, part (b) of Proposition 1 asserts that an equal steady state must involve a rate of return on human capital equal throughout to the return on financial capital, so that the set of investment options reduces (in terms of pecuniary costs and returns) to a Becker-Tomes benchmark.

Part (c) is the core of Proposition 1, and a central observation of the paper. It establishes that an unequal steady state must exhibit a wage function that has two "phases". In the first phase, which is valid for occupations that involve "low" training costs, the rates of return on human and financial capital coincide, just as they did for an equal steady state. In the second phase — and this is the deeper result — the wage function is fully characterized by a differential equation. That equation generates a rate of return which strictly exceeds that of financial bequests for (almost) all occupations with "high" training costs. In particular, the equilibrium investment frontier as viewed by households *must display an endogenous nonconvexity.*

For families that inhabit occupations with "low" training costs, financial bequests allow full equalization of wealth. But the steady state must display persistent inequality across the "high-end" ocupations. (The proposition makes precise what is meant by "high" and "low" training costs.)

The fundamental asymmetry is that given conditions [R] and [E], strict *convexity* of the investment frontier is never a steady-state outcome.

Proposition **1.** *Assume* [R], [E] *and* [LP] *hold.*

(a) *Every steady state has an equivalent representation with a continuous wage function.*

(b) *In an equal steady state, this equivalent representation may be described as follows: there exists* $w \geq 0$ *such that*

$$w(x) = w + (1 + r)x \tag{5}$$

for all x*. In that steady state, all families attain a common wealth of* $\Omega(w, r)$*, and*

$$X \leq \frac{\Omega(w, r) - w}{1 + r}, \tag{6}$$

where X *is the highest training cost across all occupations.*

(b) *In an unequal steady state, the equivalent representation may be described as a two-phase wage function: there exists* $w \geq 0$ *and* $\theta \in [0, X)$ *such that for all occupations* h *with* $x(h) \leq \theta$*,* (5) *holds:*

$$w(h) = w + (1+r)x(h).$$

Families that choose any of these occupations at any date all attain a common wealth of $\Omega(w, r)$ *that is precisely equal to* $w(\theta)$*.*

On the other hand, for all occupations h *with* $x(h) > \theta$*,* $\mathbf{w}$ *and* $\mathbf{x}$ *are connected via the following differential equation:*

$$w'(x) = \frac{U'(w(x) - x)}{V'(w(x))}. \tag{7}$$

with endpoint constraint that the wage at cost θ *equals* $w(\theta) = w + (1+r)x(h^*)$*.*

Families in such occupations attain a wealth that is strictly greater than $\Omega(w, r)$*, and the marginal rate of return to these occupations,* $w'(x)$*, strictly exceeds* $1 + r$ *almost everywhere.*

The outline of the argument is as follows. Since (almost) every training cost must have at least one inhabited occupation, the marginal rate of return on training costs must be at least r everywhere. In other words, the presence of human capital allows parents to transfer wealth to their children at (weakly) lower cost than in a Becker-Tomes benchmark where only financial bequests are possible. Hence every family must attain at least the wealth $\Omega(w, r)$ that would have arisen in the latter context (corresponding to the flow wage w that is available to every generation even in the absence of any educational investment). In an equal steady state, the rate of return on human capital at all levels is exactly r, so that the set of investment options is exactly the same as in a Becker-Tomes benchmark with stationary (w, r). Therefore all families attain precisely the wealth $\Omega(w, r)$.

Finally, equation (6) must hold in such a steady state, because the occupation with the highest training cost X must have a wage of $w + (1+r)X$, and must be willingly chosen by *some* family.

In an unequal steady state it is trivial to see that the rate of return to *some* occupation must then exceed r. If not, the set of investment options would be just as in a Becker-Tomes benchmark with financial bequests alone, and wealth equality is then the only possible long-run outcome.

What is more subtle is the exact form the wage function must take. The proposition claims that the wage function has two phases. For low-end occupations, financial and human rates of return coincide and the wage function is linear. This phase has an endogenous delineation: any occupation with training cost below the steady state financial bequest in the Becker-Tomes benchmark with (w, r) must generate a rate of return of exactly r. (Note well that w, the lowest wage, is endogenous.)

For occupations with training costs that exceed the Becker-Tomes steady state bequest, the rates of return must be high enough to induce willing settlement. This requirement creates at least a local nonconvexity of the steady state wage function: the marginal rates of return to such occupations will *exceed* r.

A closer inspection of the differential equation (7) reveals that the shape of the wage function in the second phase relies entirely on preferences. To be sure (and as we shall see more clearly below), the existence and range of this phase will depend, among other things, on the technology. For a large class of preferences, the wage function exhibits a "global" nonconvexity, in the sense that the marginal rate of return rises monotonically with training costs beyond θ.

OBSERVATION **3.** *Consider either the constant elasticity case with $U(c) = (c^{1-\sigma} - 1)/(1-\sigma)$, $\sigma > 0$, or the case of exponential utility $U(c) = -\exp(-\alpha c), \alpha > 0$, and $V \equiv \delta U$. Then the marginal rate of return on occupations monotonically increases with training cost beyond the boundary θ described in Proposition 1.*

In the case of exponential utility, the wage function takes the form (for $x > \theta$):

$$w(x) = \frac{1}{\delta\alpha}\exp(\alpha x) + \Omega(w, r) - \frac{1}{\delta\alpha}\exp(\alpha\theta). \tag{8}$$

In the constant elasticity case $w'(x)$ monotonically increases to a finite asymptote that strictly exceeds $1 + r$.

This description stands the traditional theory on its head. That theory presumes — usually by assumption — that the rates of return to human capital must be *declining* in training cost (see, for instance, Loury (1981) and Becker and Tomes (1986)). Therefore the poorer families make all the human capital investment, and once families are rich enough so that the marginal return on human capital falls to the constant rate assumed for financial capital, all other bequests are financial.

In contrast, Proposition 1 is stated in a context in which the relative earnings of different occupations are allowed depend on the occupational distribution, for which there is considerable empirical evidence (e.g., Katz and Murphy (1992)). The proposition then asserts that the theory *endogenously* generates rates of return that run counter to the assumptions made in the literature. Financial bequests are made at the low end, while "occupational bequests" carry a higher rate of return and are made by richer families. If a rich set of occupations is essential, the concavity of returns to human capital is *never* an equilibrium outcome in any unequal steady state.

As an aside, we note that we haven't yet provided conditions for a steady state to be unequal, but will soon proceed to an analysis of this question.

It is important to see that the predicted shape of steady state wage functions depend fundamentally on the limited persistence assumption. If several occupations are essential in steady state, then to sustain the "wealth-spreading" necessitated by those occupations and to counteract convergence, wage functions must display a nonconvexity. A dynastic model, such as the one studied in Mookherjee and Ray (2003), fails [LP] (see Section 4) and cannot deliver this property of wage functions in steady state.

Are our predictions counterfactual? We think not. First, it is well known from inequality decomposition studies that earnings inequality accounts for most of overall income inequality. For instance, Fields (2004) summarizes observations from several studies, writing that that "labor income inequality is as important or more important than *all other income sources combined* in explaining total income inequality".

Second, there is evidence that *within* the class of financial bequests, which are admittedly large for rich families, *intentional* bequests are not important. For instance, Gokhale *et* *al* (2001) argue that most financial bequests in the US economy are unintentional, the result of premature death and imperfect annuitization. In the iso-elastic example, this would correspond to the case with ρ below unity. Our theory then predicts that there are no *intentional* bequests anywhere in the wealth distribution, so human capital differences entirely account for all inequality, perhaps supplemented by unintended financial bequests (which we do not formally model).[25]

Finally, it is important to remember that by "occupations", we mean not just human capital but every productive activity that is inalienable. This includes human capital but it is certainly not restricted to it. In particular, it is possible to view large financial bequests observed at the top end of the distribution as a form of occupational investment by parents, in the form of transfer of ownership or control of (partly inalienable) business activities. This interpretation is pursued further in Section 9.

6. Existence and Uniqueness of Steady States

6.1. **Existence.** Having gained an understanding of the structure of steady states in this model, we can now provide conditions on the technology and preferences that guarantee the existence of a steady state.

Our definition of a steady state includes the requirement that output must be positive, so that existence is typically nontrivial. Proposition 1 informs us that a steady state must assume a particular form. In fact, it is easy to see that *given* some baseline wage w for unskilled labor, that proposition fully pins down the wage function. The only scope for variation lies in w. It comes as no surprise, then, that the existence of a (nondegenerate) steady state depends on the economy being productive enough to sustain positive profit at one of these conceivable wage functions.[26]

[25]For a model of unintended bequests arising from uncertain life span, see, e.g., Fuster (2000).

[26]There is a family resemblance here to the existence of steady states in the multisectoral optimal growth model, which requires a "productivity condition" (see Khan and Mitra (1986)). Fixed point arguments that neglect such a productivity condition, as in Sutherland (1970), are bound to fail because they do not eliminate the trivial outcome with zero output and wealth all around.

Of course, that isn't enough. Proposition 1 describes what a steady state necessarily looks like, but is silent on the question of whether such a description is indeed *sufficient* for all the steady state conditions. This, too, will need to be addressed.

We proceed, then, by searching for a steady state using the features described in Proposition 1. To this end, we describe the family of all *two-phase wage functions.* Start with a given baseline wage of w, and set $w(x) = w + (1+r)x$ for all training costs no greater than

$$\theta(w) \equiv \min\{\frac{\Omega(w,r) - w}{1+r}, X\}.$$

(This corresponds to the old threshold θ used in Proposition 1, but now we make the dependence on w explicit.) For occupations with higher training costs — if any — the wage function is set to satisfy the differential equation (7):

$$w'(x)) = \frac{U'(w(x) - x)}{V'(w(x))},$$

with the endpoint constraint that the wage for training cost $\theta(w)$ equals $w + (1+r)\theta(w)$, or equivalently, $\Omega(w,r)$.

This procedure generates a unique wage function corresponding to any choice of w. Repeat this procedure for every $w \geq 0$: we now have the entire two-phase family, with either one of the phases conceivably degenerate.[27]

Given what we know already, the following "productivity" condition is necessary for the existence of a steady state with positive output:

[P] Unit costs $c(r, \mathbf{w})$ are less than or equal to 1 for some two-phase wage function.

It is easy enough to rewrite [P] as a productivity condition. For instance, if the production function is written as $Af(f, \boldsymbol{\lambda})$, where A is some Hicks-neutral productivity parameter, [P] states simply that A is large enough.

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[27]If $\theta(w) = X$, the second phase is degenerate. Remember that at $w = 0$, $\Omega(0,r)$ is the *limit* of $\Omega(w,r)$ as $w \downarrow 0$. It may or may not equal 0, which is always trivially a limit wealth. If it is, the first phase is degenerate.

As the following proposition reveals, condition P (in conjunction with the other maintained assumptions) is also sufficient for the existence of a steady state with positive output.[28]

Proposition **2.** *Under* [R], [E]*, and* [LP]*, a steady state with positive output exists if and only if* [P] *is satisfied.*

Condition [P] isn't at all difficult to verify, one way or the other. As an example, suppose that each training cost x corresponds to a unique occupation (so name it x as well), and that the production function takes the Cobb-Douglas form

$$\ln y = (1-\alpha)\ln k + \int_0^X \alpha(x)\ln(\lambda(x))dx + \ln A,$$

where A is a productivity parameter, $\alpha(x) \geq 0$ and $\int \alpha(x)dx = \alpha \in (0,1)$. Then it is easy to see that for any wage function $\mathbf{w}$,

$$\ln c(r,\mathbf{w}) = (1-\alpha)[\ln r - \ln(1-\alpha)] + \int_0^X \alpha(x)[\ln(w(x)) - \ln(\alpha(x))]dx - \ln A.$$

The verification of [P] therefore simply entails the choice of a wage function that minimizes $\int \alpha(x)w(x)$, and then checking whether the resulting expression above is nonpositive.

6.2. **Uniqueness.** As already explained in the Introduction and Section 2, the question of uniqueness is one of great conceptual import. We prove that there can be no more than one steady state.

Proposition 1 already takes a significant step towards this result by establishing that any steady state wage function, or at least its continuous equivalent representation, must lie in the two-phase class. As described in detail earlier, it is linear with return r over a range of training costs, and then displays a marginal rate of return that strictly exceeds r. In a broad class of cases (see, e.g. Observation 3), this marginal return can be shown to be

[28] If [P] fails, it is easy to construct a steady state with zero output: simply construct the two-phase wage function that starts from a baseline wage of zero, and place all individuals in the zero-wage occupation.

ever-increasing, though it will usually possess a finite asymptote. Finally, the training-cost threshold separating the two phases precisely corresponds to the Becker-Tomes limit bequest with wage equal to the lowest wage along this function.

To be sure, the same economic fundamentals are consistent — at least in principle — with several wage functions drawn from this two-phase class. But this is what the uniqueness proposition rules out.

PROPOSITION **3.** *Assume* [E], [R], *and* [LP]*. Then, apart from equivalent representations which change no observed outcome, there is at most one steady state.*

As already discussed, this uniqueness proposition has far-ranging implications. Apart from Mookherjee and Ray (2003), to be discussed in more detail below, this observation has gone unnoticed in the literature because the literature typically concentrates on a sparse set of occupations (usually two, as in Galor and Zeira (1993) or occasionally three, as in Banerjee and Newman (1993)). In such cases multiplicity is indeed endemic, but once the set of occupations expands such multiplicity must shrink. We reiterate that the expansion of the set of occupations does *not* convexify the set of choices. Indeed, as Proposition 1 takes pains to explain, equilibrium *non*convexity is the rule rather than the exception.

Following up on this point, our uniqueness proposition does not rule out the path-dependence of economic fortunes for *individual* families. The identities of those who inhabit the different occupational slots is up for grabs and may — will — depend on historical accident. But their *numbers* cannot.

Proposition 3 is a substantial extension of the uniqueness theorem in Mookherjee and Ray (2003) to a context in which financial capital co-exists with human capital. Indeed, given the simplified context of our model,[29] the uniqueness result of Mookherjee and Ray (2003) can be seen very easily and intuitively.

[29] Apart from the central difference of financial bequests, there are two differences between our model and that of Mookherjee and Ray (2003). First, they use a nonpaternalistic bequest motive and work with value functions. Second, training costs are endogenously determined in their model. However, these differences are minor and can be readily accommodated.

Imagine reworking Proposition 1 by imposing the additional constraint that no financial bequests are permitted. One would reasonably suppose, then, that the first phase of the two-phase function would disappear, and that any steady state wage function must be governed by the differential equation (7) *throughout*. Now it is easy to see why there can be only one such wage function. If we begin at two different initial conditions and apply (7) thereafter, the two wage trajectories cannot cross — a well-known property for this class of differential equations. In short, if there are two steady state wage functions, *one must lie entirely above the other*. But now we have a contradiction, for two wage functions ordered in this way cannot *both* serve as bonafide supporting prices for profit maximization. We obtain uniqueness when there are no financial bequests.

While this serves as some intuition for the result at hand, different considerations emerge when financial bequests are permitted. Now crossings of the two putative steady state wage functions cannot be ruled out by taking recourse to uniqueness theorems for differential equations. After all, the behavior of the wage functions is *not* governed throughout by (7); a nontrivial "first phase" makes an appearance. Instead, the formal proof must rely on behavioral arguments, based on household optimization, to rule out such crossings.

7. Inequality

We are now in a position to answer the the first key question that started this paper: under what conditions does the steady state of the model involve inequality rather than equality? In the language of proposition 1, when must the second phase of the two-phase wage function necessarily be nonempty?

A simple preliminary exercise lays the groundwork for a complete characterization of this question. This exercise concerns the production technology alone and has nothing to do with preferences.

Consider the class of all linear wage functions of the form $w(x) = w + (1 + r)x$ defined on *all* of $[0, X]$, parameterized by $w \geq 0$.

Observation **4.** *Assume* [P]*. Then there is a unique value of* w *— call it* a *— and a corresponding linear wage function* $\mathbf{w}^*$ *with* $w^*(x) = a + (1+r)x$ *for all* x *— such that* $c(r, \mathbf{w}^*) = 1$.

Now a isn't an explicit parameter of our model. But for all intents and purposes it is an exogenous primitive. To compute a all one needs is a knowledge of the production function.

As an example, recall the Cobb-Douglas case studied in Section 6.1, in which each training cost corresponds to a single occupation: Cobb-Douglas form

$$\ln y = (1-\alpha)\ln k + \int_0^X \alpha(x)\ln(\lambda(x))dx.$$

Using the same logic as in that case, it is easy to see that a must solve the equation

$$(1-\alpha)[\ln r - \ln(1-\alpha)] + \int_0^X \alpha(x)[\ln(a + [1+r]x) - \ln(\alpha(x))]dx = 0,$$

provided that condition P holds.

We are now in a position to state our central result concerning persistent inequality.

Proposition **4.** *Under* [R], [E], [LP]*, and* [P]*, the unique steady state is unequal if and only if*

$$\Omega(a, r) < a + (1+r)X. \tag{9}$$

We shall refer to the inequality (9) as the *widespan condition.* It is made up of two parts. As we noted above, the parameter a is entirely a feature of the production technology and the range of the occupational structure. On the other hand, the Becker-Tomes limit wealth map Ω is entirely a feature of preferences. The widespan condition states that Becker-Tomes limit wealth — commencing from a — is *not* enough to "span" the entire range of occupations: it (net of a and discounted by the interest rate) is smaller than the span X. Proposition 4 declares that in all such cases, the steady state must exhibit persistent inequality.

One should be careful enough to note that (9) may not prescribe a unique threshold for the span X. Such an interpretation is suggested by the rewriting of the condition as

$$X > \frac{\Omega(a, r) - a}{1 + r},$$

but observe that both a and X depend on the training cost technology. To make this a more precise, consider economies that are identical in all respects except for their training cost functions, which are drawn from an ordered family (all starting at 0 for some occupation). Such an ordered family may be parameterized by the highest training cost X.

In this class, it is very easy to see that a depends negatively on X. Therefore provided that $\Omega(w, r) - w$ is nondecreasing in w, we do generate a single-threshold restriction on span: "there is X^* such that widespan holds if and only if $X > X^*$". This will be true, for instance, for isoelastic preferences: see (10) and the accompanying discussion below. More generally, though, widespan must hold for all training costs large enough (even though (9) may not imply a single threshold for X).

Uniqueness plus widespan tells us that there is just one steady state, but it must treat individuals unequally. Conversely, if widespan fails, there is convergence to a common level of wealth for all families. The proposition asserts, therefore, that whether the disequalization or the equalization view of the market is relevant depends on whether or not the widespan condition is satisfied.

Observe that under widespan, the market *must* act to separate identical or near-identical families, as described informally in Section 2. Long-run equality is not an option. However, this paper does not contain an explicit account of dynamic equilibria from non-steady-state initial conditions.

The discussion following the statement of Proposition 3 continues to be relevant here. There is no history-dependence "in the large", as the steady state is unique. But just where an individual family will end up in that distribution profoundly depends on the distant history of that family.

The strength of the characterization result in Proposition 1 is such that it renders the proof of this proposition almost trivial. See Appendix.

8. Applications and Implications

To illustrate the implications of the span condition in Proposition 4, it is useful to invoke the example of an iso-elastic utility function.

Recall, in particular, equation (4) and the discussion following it. If $\delta \leq \frac{1}{1+r}$, (9) is always satisfied and an equal steady state never exists. There are effectively no financial bequests in the limit, so the model reduces to the disequalization model in which financial bequests are not allowed. If, on the other hand, $\delta \geq 1 + \frac{1}{r}$ then $\Omega(a, r) = \infty$ and (9) fails. Financial bequests overwhelm any inequality arising from the need to provide occupational choice incentives, and an unequal steady state cannot exist.

In the intermediate case in which δ is neither too large nor too small, (4) tells us that

$$\Omega(a, r) = \frac{\rho}{1 - r(\rho - 1)} a,$$

where $\rho \equiv [\delta(1 + r)]^{1/\sigma}$, so that the widespan condition reduces to

$$\frac{\rho}{1 - r(\rho - 1)} a < a + (1 + r)X.$$

Rearranging, we obtain the following version of widespan:

$$X > a \frac{\rho - 1}{1 - r(\rho - 1)} \tag{10}$$

We now describe effects of varying parameters of the model, which are relevant to explaining cross-country differences, or effects of technological change.

(a) *Differences in TFP Levels.* Suppose we compare two countries which differ only in their levels of total factor productivity (TFP). Then for any common value of X, the poorer country has a lower value of a, implying that it is more prone to disequalization. Intuitively, the lower level of wages reduces the intensity of the parental bequest motive: they are less willing to undertake the educational investments for high-end occupations.

The resulting shortage of people in high-end occupations causes a rise in the skill premium. This elicits the requires supply into high-end occupations, but makes wealth inequality more acute.

Technologically poorer countries are therefore more prone to disequalization.

Of course, this argument is based partly on the assumption that the range of training costs X is unaffected by wages. However, it is easy to incorporate this extension under the plausible assumption that both human and physical inputs enter into production. Then X lower in the unproductive country, but not by the same factor as a. This argument is obviously reinforced if poorer countries also possess a less productive educational technology.

(b) *Differences in TFP Growth Rates.* While TFP-related differences in poverty are positively associated with disequalization, higher *growth* may be positively related to it as well. For instance, if growth (from Hicks-neutral technical progress) causes all wages and costs to grow at a uniform rate, then — all other things being equal — the level of desired bequests will be dulled, raising the likelihood of disequalization.[30]

To the extent that poorer countries grow faster owing to a "catch-up" phenomenon in technology, the widespan condition is therefore more likely to hold on two counts: higher poverty *and* higher growth. Of course, the net result is ambiguous if subsequent growth isn't positively correlated with initial poverty.

(c) *Changes in Interest Rates.* A change in the rate of return to capital has subtle effects. When r rises, ρ also goes up. Both these effects work against the widespan condition, by raising the rate of return to financial bequests. So a first cut at this issue would suggest that an increase in the global rate of return to physical capital tends to be equality-enhancing. However, there is the possibility that a may be lowered by the increase in r. This effect runs in the opposite direction, and a full analysis is yet to be conducted.

[30]We omit a formal demonstration of this assertion, which proceeds by deriving an equivalent of the widespan condition (10) in the presence of neutral technical progress.

(d) *Reliance on Physical Capital* Now let us compare economies with differing degrees of mechanization, i.e., reliance on physical capital *vis-a-vis* human capital in production. One simple way to do this is to suppose that final output is produced via a nested function

$$y = Ak^{\alpha}m^{1-\alpha},$$

where m is a composite of the occupational inputs: e.g., an intermediate good "produced" by workers. Then greater mechanization corresponds to a rise in α. Setting the marginal product of capital to the interest rate r, we obtain

$$\frac{k}{m} = \left(\frac{A}{r}\right)^{\frac{1}{1-\alpha}},$$

so that the indirect "reduced-form" production function is linear in m:

$$y = Bm,$$

where

$$B = A\left(\frac{A}{r}\right)^{\frac{\alpha}{1-\alpha}}.$$

Notice that B essentially prices the composite in terms of the final output. If B goes down for some reason, then $w(0)$ will decline. So a reduction in B, other things being equal, will contribute to a greater likelihood of disequalization. Whether B goes up or down with α depends on the ratio of A (TFP) to r.[31] In relatively "unproductive" economies in which A is small, an increase in physical capital intensity lowers B, making inequality more likely. The opposite is the case in "productive" economies in which A is large. We thus obtain an interesting answer to a classic question in the theory of distribution: the impact of greater mechanization in production on long-run inequality.

(e) *Wider Product Variety.* Wider occupational spans may be the outcome of introduction of new goods and services, owing to technological change. The production of new goods and services such as information and communication technology creates an entirely new set of occupations. Such occupations are likely to require high levels of education

[31]The intuitive reason why this is the appropriate comparison is that A becomes the productivity of capital in the limiting case when $\alpha = 1$.

and training, which may be thought of as an increase in the span of occupations and associated training costs. Unlike the parameterization used in Proposition 4, such changes involve an increase both in X and in the productivity of the technology. In terms of (10), both X and a tend to rise and the net effect depends on the ratio of these two variables.

We have not yet analyzed this application in detail, but it is clear that the effect of wider product variety on inequality is a very important question.

(f) When (9) holds, the rising rate of return to occupations (as captured by Observation 3) is an important prediction of the theory. We have discussed this already, but it bears repetition that this *derived* property of the model contradicts the *assumed* properties of human capital in Becker and Tomes (1986) and others. Empirical research will take us closer to settling this issue.

9. Extensions

We now discuss the implications of extending our model or dropping some important assumptions made so far. Further details are available in the working-paper version of this paper (Mookherjee-Ray (2006)).

9.1. **Investment Indivisibilities.** What happens when we suppose there are a finite set of occupations? Then occupational investments are subject to an indivisibility: between many pairs of occupations with varying training costs, there will be no "middle" option with intermediate costs. An appropriate extension of the characterization of equal and unequal steady states continues to apply, as well an existence result and the widespan condition for existence of an equal steady state.

The major difference us that steady states may no longer be unique: Proposition 3 will no longer apply. For instance, when the widespan condition fails, both equal and unequal steady states can co-exist. History will determine which steady state the economy converges to. This is the basis for a market view of history (there are others, that we do not address here[32]) espoused by several authors; see Introduction. The indivisible case is, however, more amenable to a study of non-steady-state dynamics.[33]

9.2. **Autarky: Endogenous Interest Rate.** Under autarky, the interest rate is endogenously determined by the condition that the capital market must clear. All propositions in the paper now apply, conditional on the interest rate.[34] Conditional on baseline w and r, the shape of wage functions and desired bequests are still described by the steady state constructed in Proposition 2. The supply of capital comes from desired bequests from those in low skilled occupations: this has to equal the demand for capital from firms at the given schedule of factor prices. The baseline wage w and interest rate r are determined by the condition of profit maximization and clearing of the capital market.

However, we find no general condition that guarantees uniqueness of the interest rate. Hence multiple steady states associated with different interest rates are now possible. Long run cross-country income differences can be explained by historical differences, resulting in different interest rates. Related forms of steady state multiplicity have been discussed in Piketty (1997) and Banerjee and Duflo (2004). Countries with a higher interest rate will involve lower unskilled wages and a higher skill premium, while comparisons of skilled wages are ambiguous.[35]. Note also that this point of view suggests

[32]See, e.g., Sokoloff and Engerman (2000), Acemoglu, Johnson and Robinson (2001), and Banerjee and Iyer (2005), who rely on political-economy explanations.

[33]In the working-paper version of this paper we obtain the interesting and novel result that even if an equal steady state exists, convergence to an unequal steady state can occur from an initial condition of perfect equality, if the economy is poor enough to start with. This is a particularly striking implication of disequalization owing to "symmetry-breaking".

[34]The existence of steady states requires two additional assumptions on underlying utility and production functions which are elaborated in the previous version of this paper. Essentially these ensure that there is excess demand for capital when r approaches -1, and there is excess supply of capital when r approaches ∞.

[35]It is possible that wages of all occupations are lower, if the interest rate is higher: this is consistent with profit-maximization

that integration of capital markets across countries will tend to promote convergence across countries.[36]

9.3. **Credit Rationing and Size Distribution of Firms.** So far, we have not allowed households to borrow at all. Moreover, the term "occupation" has been used to describe a particular form of human capital. We now address an alternative interpretation, in which "occupations" are firms of different sizes, the ownership of which cannot be fully diversified in a stock market. Firms rely on self-finance or an imperfect capital market to fund their set-up costs. We explain how our framework can be extended to incorporate models of distributional dynamics through the credit market rather than human capital (as in Aghion and Bolton (1997), Piketty (1997), Lloyd-Ellis and Bernhardt (2000), Matsuyama (2000, 2003) or Banerjee and Duflo (2004)).

As before, an individual starts life with wealth inherited from her parents, but now starts up one of several firms with varying fixed costs (interpret "labor" as starting a firm with zero fixed costs). The individual faces borrowing constraints depending on the size of inherited wealth, and so is subject to an upper bound on the size of her firm. Firms of different sizes produce intermediate goods that are imperfect substitutes for one another, with perfect substitutes a limiting case. Intermediate goods combine to produce a single consumption good according to a production function. An assumption on the "essentiality" of different intermediate goods in the production of the final good is then a convenient way of obtaining diversity in firm size.

[36]This is in contrast to Matsuyama (2004) and Bajona and Kehoe (2006), where the opposite can happen. These papers assume there are just two factors of production: capital and labor, and a conventional neoclassical production function. Under autarky, poorer countries with less capital obtain a higher rate of return to their investments, and thus grow faster. With capital market integration (or factor price equalization owing to product market integration), the rate of return becomes equal across all countries. Interest rates decline in poor countries and rise in rich countries, thus shutting down the key force towards convergence. These results do not apply in our model owing to the endogenous nonconvexities in returns to investment, associated with the heterogeneity of different forms of human capital.

The model developed in the preceding sections represented the capital market as a frictionless transaction between lender-households and borrower-firms producing final consumption goods. We now need to focus on frictions in the capital market. To do so, we simplify by assuming that production of the final good from intermediates does not require any physical capital at all. Physical capital is required only in the production of intermediate goods, and we focus on the capital market for these sectors.

The CRS production function for the final consumption good is given by $y = f(\boldsymbol{\lambda})$, where $\boldsymbol{\lambda}$ is a firm size distribution over the set of intermediate good sectors $\mathcal{H}$. We shall refer to a sector by its setup capital requirement k, so the "training cost" function is just $x(k) \equiv k$. Perfect competition in the final good sector determines returns $w(k)$ to the production of different intermediate goods: $w(k)$ depends on the firm size distribution λ via marginal products.

An individual with inheritance I is subject to a borrowing constraint: she can invest in a firm of size k provided $k \leq \Gamma(I; r)$, where for each r, Γ is increasing in I with $\Gamma(I; r) \geq I$.[37] Because parents care for child wealth, one can suppose without loss of generality that a parent allocates her overall bequest I between physical capital k and financial assets b. [The child would make the same choice, given I.] Within the region specified by the borrowing constraint, b can indeed be negative: for given I, the parent can bequeath any combination (b, k) such that $b = I - k$ and $k \leq \Gamma(I; r)$. When b is negative, it corresponds to lending. As it will turn out, poorer households will lend to richer households, so the capital market enables a widening of the distribution of firm sizes.

A parent of wealth W that anticipates an interest rate r and firm size distribution $\boldsymbol{\lambda}$ for the next generation then faces the following choice: select an asset portfolio (b, k) for its child to maximize

$$U(W - b - k) + V((1 + r)b + w(k; \lambda)) \tag{11}$$

[37]This borrowing limit is the only capital market imperfection; the borrowing and lending rates of interest are assumed to be the same. Several microfoundations for this kind of imperfection are available in the literature: see e.g., Ghosh, Mookherjee and Ray (2001).

subject to the constraint

$$b \geq \Gamma^{-1}(k; r) - k, \tag{12}$$

which is a restatement of the household optimization problem (1), with a more general version of the borrowing constraint.

The definition of a competitive equilibrium is slightly different, owing to the requirement of clearing of the capital market. It is a sequence $(r_t, \boldsymbol{\lambda}_t, w_t(k))$ such that: (i) a parent i with wealth $W_t(i) \equiv (1 + r_t)b_t(i) + w_t(k_t(i))$ derived from its asset portfolio $(b_t(i), k_t(i))$ selects a portfolio $(b_{t+1}(i), k_{t+1}(i))$ for its child to maximize (11) with $r = r_{t+1}$, $w = w_{t+1}$; (ii) $\boldsymbol{\lambda}_t$ is generated by distribution of $k_t(i)$, and $\int b_t(i)di = 0$; (iii) $w_t(k) = w(k; \boldsymbol{\lambda}_t)$.

With relatively minor modifications, our previous analysis applies largely unchanged to this context, as explained in Appendix F of Mookherjee and Ray (2006). In this setting we obtain a more natural interpretation of inequality at the top end of the distribution, compared to the case of pure human capital. All bequests are "financial" in this world, and the largest financial bequests are observed at the top rather than bottom end of the distribution. The right notion of "occupation" thus seems to involve more than just human capital or acquisition of skill: it embraces also the wherewithal to enter sectors of the economy with large start-up costs.

10. Conclusion

This paper addresses two central questions in the theory of income distribution:

[I] Do competitive markets equalize or disequalize wealth allocations?

[II] Does history matter? Are the same economic fundamentals consistent with multiple steady states?

We study a model of intergenerational bequests which allows for both financial bequests as well the choice of a rich variety of occupations. A fundamental postulate of the model is that occupational inputs are *imperfect* substitutes, so that factor prices are endogenously determined. At the level of an individual household, occupational investments may be fine-tuned to an arbitrary degree, provided that there is rich variation in occupations and training costs. But the returns to these occupations are endogenous, and the equilibrium of the market will determine whether households face a convex or nonconvex investment technology.

We provide a complete characterization of steady states. We show that if there is inequality in steady state (so that question I has been answered affirmatively), then the steady state wage function must be linear over a section and convex over others, so that each family must face an investment nonconvexity. This finding is at sharp odds with existing literature that simply assumes the opposite: that the rate of retrun to human capital must exhibit diminishing returns.

Our characterization permits us to move on to the two central questions. We prove that with a rich set of occupations, the steady state must be unique. There may (and generally will) be path-dependence at the level of dynastic choices, but the *overall* distribution of outcomes must be independent of history. This provides an interesting negative answer to question II, and in this way challenges a literature that is predicated on steady state multiplicity.

The second principal goal of this paper is to address question I: we extend and unify three seemingly different views of market-driven inequality: *equalization*, embodied in traditional theories of income distribution, *disequalization*, central to the recent endogenous inequality literature, and *neutrality*, that either can happen depending on historical conditions. We show that a fundamental condition, which we call the widespan condition, allows us to predict whether the steady state must involve inequality or not.

The widespan condition draws attention to an aspect of technology that has received little attention in the literature: the range or "span" of occupational structure. Whether equality or inequality results from market mechanisms depends on this key parameter representing the extent of occupational diversity, relative to the strength of bequests.

The theory has several potential applications and implications.

We end with a final comment on question II. Our uniqueness result stems from the assumed richness in occupational investments. Whether or not such richness exists therefore deserves to be studied empirically. If the assumption is valid, theories of macroeconomic history dependence will have to be based either on interest rate multiplicities rooted in the lack of capital market integration, or political economy channels, rather than market-based occupational choice mechanisms.

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Appendix

Proof of Observation 1. Let W_t denote the wealth of a typical member of generation t. We claim that if $W_{t+1} < W_t$, $W_{s+1} \leq W_s$ for all $s > t$. Suppose not; then there exist dates τ and s with $s > \tau$ such that (a)$W_{\tau+1} < W_\tau$, (b) $W_{s+1} > W_s$, and (c) $W_m = W_{\tau+1} = W_s$ for all intermediate dates $\tau + 1 \leq m \leq s$ (this last requirement is, of course, vacuous in case $s = \tau + 1$). But then a *strictly* higher wealth (W_τ compared to W_s) generates a strictly lower descendant wealth ($W_{\tau+1}$ compared to W_{s+1}), which contradicts a familiar "single-crossing" argument based on the strict concavity of the utility function. Therefore dynastic wealth converges in this case.

Next suppose that $W_{t+1} > W_t$. Then a reversal of the same logic implies that family wealth is nondecreasing across generations. ■

Lemma **1.** *In the Becker-Tomes benchmark,* $\tilde{W} - W$ *has exactly the same sign (including equality) as*

$$(1+r) - \frac{U'(\frac{rW+w}{1+r})}{V'(W)}. \tag{13}$$

Proof. Simply recall the first-order conditions (2) and the concavity of U and V. ■

Proof of Observation 2. We first show that under (H),

$$U'(\frac{rW+w}{1+r})/V'(W) \text{ is increasing in } W, \tag{14}$$

for all $w > 0$ and $r > -1$. It is sufficient to show that

$$V'(W)\frac{r}{1+r}U''(\frac{rW+w}{1+r}) - U'(\frac{rW+w}{1+r})V''(W) > 0$$

which is equivalent to

$$-\frac{U''(\frac{rW+w}{1+r})}{U'(\frac{rW+w}{1+r})}\left(\frac{r}{1+r}\right) < -\frac{V''(W)}{V'(W)}.$$

Given (H), this reduces to the condition that

$$\frac{r}{1+r} < \frac{\alpha + \beta[\frac{rW+w}{1+r}]}{\alpha + \beta W},$$

which is always true *unless* $w = 0$ *and* $\alpha = 0$. This immediately verifies [LP], barring these exceptional cases.

If $w = 0$ and $\alpha = 0$, then $U'(\frac{rW+w}{1+r})/V'(W)$ is unchanging in W, so that [LP] is verified for all but one value of r. ■

To prepare for the proofs of the remaining propositions, we record several lemmas, and we presume (often implicitly) that [R], [E] and [LP] apply where needed.

LEMMA **2.** *In the Becker-Tomes benchmark, for every* $w > 0$ *and* $r > -1$*:*

(a) *If* $w \leq W < \Omega(w, r)$*, then* $W < \tilde{W}(W; w, r) \leq \Omega(w, r)$*. If* $W > \Omega(w, r)$*, then* $\Omega(w, r) \leq \tilde{W}(W; w, r) < W$*.*

(b) $\Omega(w, r)$ *is nondecreasing in* w*.*

(c) $\Omega(w, r)$ *is continuous at every* (w, r) *at which it is finite.*

Proof. Part (a) follows simply from the fact that $\tilde{W}(W; w, r)$ is nondecreasing in W. If, for instance, $\tilde{W}(W; w, r) \leq W < \Omega(w, r)$, or if $\tilde{W}(W; w, r) > \Omega(w, r) > W$, limit wealth starting from W can never equal $\Omega(w, r)$. This proves the first assertion in part (a); the second assertion follows in similar fashion.

Part (b). We claim first that $\tilde{W}(W; w, r)$ is nondecreasing in w. To show this, it suffices to assume that $\tilde{W} > w$, but then it must be the case that

$$U'\left(W - \frac{\tilde{W} - w}{1 + r}\right) \leq (1 + r)V'(\tilde{W}). \tag{15}$$

Now suppose that w goes up to w', but $\tilde{W}$ falls to $\tilde{W}'$. The new situation must therefore have strictly positive consumption for the parent, so that the new first-order condition has the opposite weak inequality:

$$U'\left(W - \frac{\tilde{W}' - w'}{1 + r}\right) \geq (1 + r)V'(\tilde{W}'). \tag{16}$$

But (15) and (16) together contradict the assumption that U and V are strictly concave.

With this claim in place, the result follows immediately from [LP].

Part (c). Suppose that $\Omega(w, r)$ is finite. By part (a), $\tilde{W}(W; w, r)$ (viewed as a function of W) must "strictly intersect the 45^0 line" at the value $\Omega(w, r)$. The continuity of $\tilde{W}(W; w, r)$ now assures us that $\Omega(w, r)$ must be locally continuous. ■

LEMMA **3.** *For any steady state wage function:*

(a) *If* h *is inhabited,* $x(h) = x(h')$ *implies* $w(h) \geq w(h')$*.*

(b) *If h is inhabited, then $x(h) > x(h')$ implies $w(h) - w(h') \geq (1+r)[x(h) - x(h')]$.*

Proof. If (a) is false and $x(h) = x(h')$ while $w(h) < w(h')$ then any parent selecting occupation h for her child would do better to select occupation h' instead. The same is true if (b) were false: the parent could switch to occupation h' for the child, combined with a higher financial bequest so as to leave the child's wealth unaffected, while increasing her own consumption. ■

As in the main text, it will often be convenient to write the wage as a function of training cost: $w(x)$. Part (a) of Lemma 3 informs us that we can certainly do this right away for inhabited training costs.

Lemma **4.** (a) *The set of all inhabited training costs — call it T — is a subset of $[0, X]$ of full measure, and $\mathbf{w}$ is continuous on T.*

(b) *There is a unique continuous extension of $\mathbf{w}$ on T to all of $[0, X]$, and it forms an equivalent representation.*

Proof. Part (a). Let T be the set of all inhabited training costs. Since a steady state must have positive output by definition, it follows from [R] and [E] that T must be of full measure. Moreover, $\mathbf{w}$ (viewed as a function of x) must be continuous on T. For if not, we can select training costs x and x' in T that are arbitrarily close, but such that their wage difference is bounded away from zero. In that case no parent would select the (almost identical) training cost with a lower wage.

Part (b). Because $\mathbf{w}$ is nondecreasing on T and T is of full measure, there is a unique continuous extension of $w(x)$ to all of $[0, X]$. Use this extension to define $\hat{w}(h)$ for all occupations, inhabited or not. That is, $\hat{w}(h) = w(h)$ if h is inhabited, and equals the continuous extension otherwise. We first claim that $\hat{w}(h) \geq w(h)$ for all h that are uninhabited. For if not, we have a contradiction in a manner similar to part (a): we can find an inhabited occupation h' arbitrarily close to h but with wages bounded below that of $w(h)$, which means that all occupiers of h' would prefer h, a contradiction.[38]

By this claim, if we replace $\mathbf{w}$ by $\hat{\mathbf{w}}$, no firm will wish to change its desired input mix (unused inputs have not become any cheaper). To complete the proof of equivalence, observe that no family occupying h' finds it strictly profitable to switch to an uninhabited occupation h once its wage has been replaced by $\hat{w}(h)$. For if this were true, then by the definition of continuous extension we can find a third *inhabited* occupation h'' with wage and training cost arbitrarily close to that of h, such that the family must therefore also find it profitable to switch from h' to h''. But this is a contradiction, since that option is already available in the going steady state. ■

[38]This argument, as well as its counterpart for (a), can be made entirely precise by showing that the preference for h over h' can be made uniform over all families, irrespective of their wealth.

In what follows we focus exclusively on this equivalent representation and call it $\mathbf{w}$ instead of $\hat{\mathbf{w}}$. Define $w = w(0)$.

Lemma **5.** *Every family attains a wealth of at least* $\Omega(w, r)$.

Proof. Let W be the steady state wealth of some family. Suppose that it incurs training cost x and leaves bequest b; then by stationarity, $W = (1+r)b + w(x)$. Now, *given* the choice of x, b must maximize

$$U(W - x - b) + V(w(x) + (1+r)b) \tag{17}$$

subject to $b \geq 0$. Defining $B \equiv b + x$, this can equivalently be written as: B is chosen to maximize

$$U(W - B) + V(w' + (1+r)B) \tag{18}$$

subject to $B \geq w(x)$, where $w' \equiv w(x) - (1+r)x$. It is obvious that the solution to this problem involves a total bequest B at least as large as the value that would obtain if the constraint $B \geq w(x)$ were replaced by $B \geq 0$, i.e., if we were in a Becker-Tomes benchmark world with w' and r. Remembering that W is also next period's wealth in the problem (18), we conclude that

$$W \geq \tilde{W}(W; w', r). \tag{19}$$

We now claim that

$$W \geq \Omega(w', r). \tag{20}$$

If this is false, then $W < \Omega(w', r)$. Combining this with (19), we may conclude that $\tilde{W}(W; w', r) \leq W < \Omega(w', r)$, but this contradicts part (a) of Lemma 2.

Now recall the definition of w' and invoke part (b) of Lemma 3 to conclude that $w' \geq w$. Use this fact and part (b) of Lemma 2 to observe that $\Omega(w', r) \geq \Omega(w, r)$. Combine this last inequality with (20) to obtain the desired result. ■

Lemma **6.** *Let* W *be any lower bound on stationary wealth across all families.*

(a) *If for any occupation* h*, we have* $w + (1+r)x(h) \leq W$*, then* $w(h) = w + (1+r)x(h)$.

(b) *The stationary wealth of any family selecting an occupation* h *with* $w + (1+r)x(h) \leq W$ *must be* $\Omega(w, r)$.

Proof. Part (a). Suppose, on the contrary, that $w + (1+r)x(h) \leq W$, but $w(h) \neq w + (1+r)x(h)$. By Lemma 3, part (b), and the fact that $\mathbf{w}$ is a continuous equivalent representation, it must be that

$$w(h) > w + (1+r)x(h).$$

so that for some $x' < x(h)$ but close to it,

$$w(x') > w + (1+r)x'. \tag{21}$$

Now, we know that there is a sequence of inhabited occupations $\{h^k\}$ such that $x(h^k) \downarrow 0$. By assumption, W is a lower bound on stationary wealth across all families, and $w + (1+r)x' < w + (1+r)x(h) \leq W$. Therefore (for k large enough) such families are almost exclusively leaving financial bequests of magnitude at least x' to their children at rate of return r. They would be better off, instead, selecting an occupation with cost x' for their children (which yields a return of strictly more than r, by (21), and supplementing the remainder with financial bequests, a contradiction.

For each family selecting occupation h with $w + (1+r)x(h) \leq W$, we have $w(h) = w + (1+r)x(h)$. Therefore the realized rate of return to *all* the choices of such a family, financial and educational, is exactly r. Define $\hat{x}$ by

$$w + (1+r)\hat{x} = W. \tag{22}$$

Once again, using Lemma 3, part (b), and the fact that $\mathbf{w}$ is a continuous equivalent representation, we also know that wages for training costs beyond $\hat{x}$ yield *no less* a return than r for all educational investments beyond $\hat{x}$. Yet these families choose (by part (a)) not to utilize such regions of educational investment. They must therefore be behaving in exactly the same way as in a Becker-Tomes benchmark world with parameters (w, r). We must conclude that their stationary wealth equals $\Omega(w, r)$. ■

Lemma **7.** *In an equal steady state, the common wealth of all families must be* $\Omega(w, r)$*, and* $\Omega(w, r) \geq w + X(1+r)$.

Proof. Let W denote the (common) wealth of each family in an equal steady state. We claim that

$$W \geq w + (1+r)X. \tag{23}$$

For if not, we know from Lemma 5, part (b) (and the fact that $\mathbf{w}$ is a continuous equivalent representation) that there are inhabited occupations with training costs x arbitrarily close to X, and that

$$w(x) \geq w + (1+r)x.$$

Consequently, if (23) were to be false, we would find inhabited occupations with $w(h) > W$. Because the wealth of families in such occupations is at least $w(h)$, this is a contradiction to equality.

It remains to prove that $W = \Omega(w, r)$. Because (23) is true, and because W is (trivially) a lower bound on stationary wealth, Lemma 6, part (b) applies to all families, and $W = \Omega(w, r)$. ■

Lemma **8.** *A steady state is equal if and only if the continuous equivalent representation of the wage function is linear:*

$$w(h) = w + (1+r)x(h) \tag{24}$$

for all occupations h.

Proof. For necessity, combine Lemma 6, part (a), and Lemma 7. For sufficiency, note that if (24) holds, then we are in a Becker-Tomes benchmark world and all limit wealths must be the same. ■

The remaining steps concern unequal steady states. Define θ by

$$w + (1+r)\theta = \Omega(w, r). \tag{25}$$

Lemma **9.** *In an unequal steady state, it must be that $\theta < X$.*

Proof. By Lemma 5, $\Omega(w, r)$ is a lower bound on stationary wealth for all families. If, contrary to our assertion, $\theta \geq X$, then part (a) of Lemma 6 applies for *every* training cost, when W is replaced by $\Omega(w, r)$. Therefore $\mathbf{w}$ satisfies (24), and the sufficiency direction of Lemma 8 implies that the steady state must be equal, a contradiction. ■

We are interested in the shape of $\mathbf{w}$ in the region $U \equiv [\theta, X]$.

Lemma **10.** *Let I be some subinterval of U such that no financial bequests are made by any family that occupies some occupation with training cost in U. Then $\mathbf{w}$ satisfies* (7) *on I.*

Proof. Fix any $x \in I$, with $x < \sup I$. For $\epsilon > 0$ but small enough, $x+\epsilon \in I$ as well. Assume provisionally that both x and $x+\epsilon$ are inhabited. Then family wealth at x (resp. $x+\epsilon$) is merely $w(x)$ (resp. $w(x+\epsilon)$). Using the two optimality conditions, one for families with wealth $w(x)$ and the other for families with wealth $w(x+\epsilon)$, we see that

$$U(w(x)-x)-U(w(x)-(x+\epsilon)) \geq V(w(x+\epsilon))-V(w(x)) \geq U(w(x+\epsilon)-x)-U(w(x+\epsilon)-(x+\epsilon)). \tag{26}$$

Now, using the fact that $\mathbf{w}$ is a continuous equivalent representation, and invoking [R] and [E], we can see that (26) actually applies to *all* x and $x+\epsilon$ in I, not just those that are inhabited.[39]

Dividing these terms throughout by ϵ, applying the concavity of the utility function to the two side terms, and the mean value theorem to the central term, we see that

$$U'(w(x) - (x+\epsilon)) \geq V'(\gamma(\epsilon))\frac{w(x+\epsilon) - w(x)}{\epsilon} \geq U'(w(x+\epsilon) - x),$$

[39]Given [R] and [E], we can approach both x and $x+\epsilon$ by a sequence of inhabited training cost pairs in I, and for each such pair the inequality (26) holds.

where $\gamma(\epsilon)$ lies between $w(x)$ and $w(x+\epsilon)$. Now send ϵ to zero and use the continuous differentiability of U and V to conclude that the right-hand derivative of w with respect to x — call it $w^+(x)$ — exists, and

$$w^+(x) = U'(w(x) - x)/V'(w(x)).$$

By exactly the same argument applied to x (greater than $\inf I$) and $x - \epsilon$, we may conclude the same of the left-hand derivative, which verifies (7). ■

Lemma **11.** *Any two-phase wage function has $w'(x) > 1 + r$ for almost all $x > \theta$.*

Proof. The continuous differentiability of U and V, and the continuity of $\mathbf{w}$ together imply that $\mathbf{w}$ is continuously differentiable in its second phase, where it follows (7). Note moreover that $w'(\theta) = 1 + r$. Therefore, if the assertion is false, there is an interval $[x_1, x_2]$, with $x_1 \geq \theta$, on which $w'(x) \leq 1+r$, while $w'(x_1) = 1 + r$ and $w(x_1) - (1 + r)x_1 \geq 0$.[40] Applying (7) at x_1 and using $w'(x_1) = 1 + r$, we see that

$$U'(w(x_1) - x_1)) = (1 + r)V'(w(x_1)). \tag{27}$$

Define $\hat{w} = w(x_1) - (1 + r)x_1$, which is nonnegative by construction. Then (27) reduces to

$$U'\left(\frac{rw(x_1) + \hat{w}}{1 + r}\right) = (1 + r)V'(w(x_1)), \tag{28}$$

By the same logic as (27), except that at x_2 we have $w'(x_2) \leq 1 + r$, we see that

$$U'(w(x_2) - x_2)) \leq (1 + r)V'(w(x_2)). \tag{29}$$

Now observe that $w(x_1) + (1 + r)(x_2 - x_1) \geq w(x_2)$ (because $w'(x) \leq 1 + r$ on $[x_1, x_2]$), or equivalently, using the definition of $\hat{w}$, $x_2 \geq \frac{w(x_2) - \hat{w}}{1+r}$. Consequently,

$$w(x_2) - x_2 \leq \frac{rw(x_2) + \hat{w}}{1 + r},$$

and using this in (29) along with the concavity of U, we must conclude that

$$U'\left(\frac{rw(x_2) + \hat{w}}{1 + r}\right) \leq (1 + r)V'(w(x_2)). \tag{30}$$

Suppose that $w(x_1) > 0$. Then (28) means that $w(x_1)$ is a positive limit wealth in the Becker-Tomes benchmark with $(\hat{w}, r)$, while Lemma 1 tells us that $\tilde{W}(w(x_2); \hat{w}, r) \geq w(x_2)$. Because $w(x_2) > w(x_1)$, this means that $w(x_1)$ *cannot* be a limit wealth starting from initial wealth $w(x_2)$. This contradicts [LP].

[40] One way to assure the existence of an interval with all these properties is to take x_1 to be the minimum of the values among those greater than θ for which $w'(x) \leq 1 + r$.

On the other hand, if $w(x_1) = 0$, then $w = 0$ and $\theta = 0$ as well, which means that $\Omega(0, r) = 0$: the limit of Becker-Tomes wealth from all *positive* initial wealths is zero when $w = 0$.[41] But this fact is contradicted by Lemma 1 applied to (30), because $\hat{w}$ is also 0 in this case. ■

If a wage function satisfies $w(x) - w(x') = (1+r)(x - x')$ for all x and x' in some interval, say that it is r-*linear* over that interval. We know, for instance, that any two-phase wage function with a nondegenerate first phase indeed r-linear over $[0, \theta(w)]$.

LEMMA **12.** *Suppose that a family in steady state, inhabiting training cost x at some date, also makes a financial bequest at that date. That is, it possesses (and bequeaths) total wealth W, where $W > w(x)$. Then* **w** *is r-linear over all $x' \geq x$ with $w(x) + (1 + r)(x' - x) \leq W$:*

$$w(x') = w(x) + (1 + r)(x' - x). \tag{31}$$

Proof. The proof is very similar to that of Lemma 6. Pick any x' with $w(x) + (1 + r)(x' - x) \leq W$. Our family is making a financial bequest of at least $x' - x$. If (31) were to fail, then by Lemma 3, part (b), and the fact that **w** is a continuous equivalent representation, we must have

$$w(x') > w(x) + (1 + r)(x' - x),$$

which means that our family would certainly be better off choosing x' instead of x and supplementing the remainder (if any) with financial bequests, a contradiction. ■

The next lemma summarizes what we know so far about an unequal steady state.

LEMMA **13.** *The continuous equivalent-representation of any unequal steady state wage function is r-linear up to θ, followed by combinations of intervals over which either the differential equation* (7) *is obeyed, or r-linearity holds.*

Proof. Combine Lemmas 10 and 12. ■

However, we now establish a stronger property:

LEMMA **14.** *In an unequal steady state, the continuous, equivalent-representation wage function must be two-phase.*

[41] Remember: this is a different statement from the one that asserts that 0 is a Becker-Tomes limit wealth when $w = 0$, which is always trivially true.

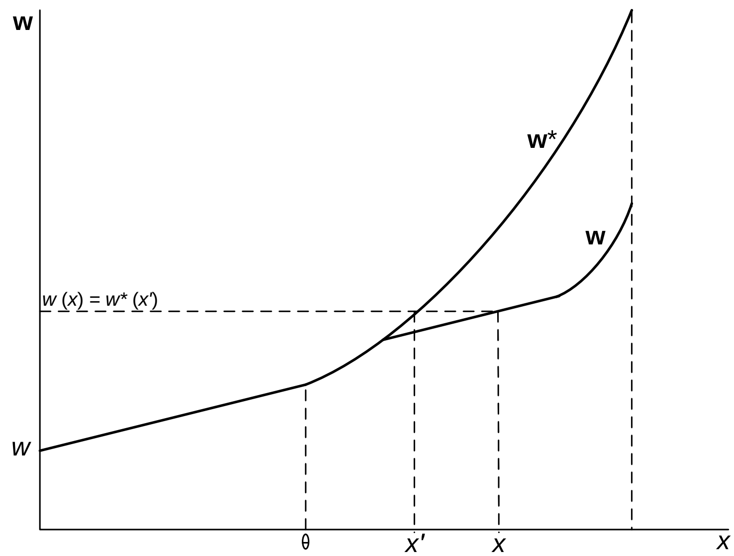

FIGURE 1. Proof of Lemma 14

Proof. Let $\mathbf{w}$ be a (continuous) steady state wage function, starting from w. Denote by $\mathbf{w}^*$ the two-phase wage function starting from the same point. We know already that the two functions coincide at least up to θ. Suppose, contrary to the assertion, that $w^*(x) \neq w(x)$ for some $x \in (\theta, X]$. Then there is some *first* r-linear segment "after" θ at which $\mathbf{w}$ departs from $\mathbf{w}^*$.

By Lemma 11, $\mathbf{w}$ must lie below $\mathbf{w}^*$ in this segment. Use Figure 1 as a guide in what follows.

Pick some inhabited x in the interior of the r-linear segment; then pick $x' \in (\theta, x)$ such that

$$w^*(x') = w(x) > 0. \tag{32}$$

Pick any family that inhabits x at any date, and has stationary wealth W. The first-order conditions for utility maximization tell us that

$$U'(\frac{rW + w}{1 + r}) = (1 + r)V'(W),$$

so that $W > 0$ is a limit wealth in the Becker-Tomes benchmark with baseline wage $w' = w(x) - (1+r)x$ and rate of return r. By [LP] applied to this benchmark, a family with starting wealth $W' = w(x) > 0$ in this benchmark world must converge to the very same limit wealth. Because $W' = w(x) \leq W$, convergence to W must require that $\tilde{W}(W'; w', r) \geq W'$. Invoking Lemma 1 and writing $w(x)$ in place of W', we must conclude that

$$(1 + r)V'(w(x)) \geq U'(\frac{rw(x) + w'}{1 + r}) = U'(w(x) - x).$$

Now recall the definition of x' from (32). Replacing $w(x)$ by the same value $w^*(x')$, replacing x by the smaller value x', and using the strict concavity of U, we see that

$$(1+r)V'(w^*(x')) > U'(w^*(x') - x'),$$

but this contradicts the fact that $\mathbf{w}^*$ satisfies (7) at x'. ■

Proof of Proposition 1. Lemma 4 establishes that there is a continuous equivalent representation to the wage function in every steady state. Lemma 7 shows that in any equal steady state, $\Omega(w, r) \geq w+(1+r)x$, where w is the lowest wage in that steady state wage function. Lemma 24 shows that the wage function must be r-linear for equal steady states. Lemma 14 shows that unequal steady state wage functions must have the two-phase property: it is r-linear up to θ, which is defined in (25), and follows the differential equation (7) thereafter. Lemma 11 shows that the second phase must exhibit a rate of return that is almost everywhere higher than r. ■

Proof of Observation 3. The differential equation (7) in the exponential utility case reduces to

$$w'(x) = \frac{1}{\delta\alpha}\exp(\alpha x) \tag{33}$$

from which the stated result follows. Applying (7) to the constant elasticity case, we see that for all $x \geq \theta$,

$$w'(x) = \frac{1}{\delta}\left[\frac{w(x)}{w(x)-x}\right]^{\sigma}. \tag{34}$$

Differentiation of this equality shows us that

$$w''(x) = \sigma\left[\frac{w(x)}{w(x)-x}\right]^{\sigma-1}\frac{w(x)-xw'(x)}{[w(x)-x]^2},$$

so that $w''(x)$ is continuous and has precisely the same sign as $w(x)/x - w'(x)$. Notice that

$$\frac{w(x)}{x} > w'(x)$$

at $x = \theta$. So $w'(x)$ increases just to the right of θ, while — using (34) — $w(x)/x$ monotonically falls. But it must be the case throughout that $w(x)/x$ continues to exceed $w'(x)$, otherwise the very changes described in this paragraph cannot occur to begin with. Therefore $w'(x)$ rises throughout, establishing strict convexity to the right of θ.

However, w' cannot go to ∞, as another perusal of (34) will readily reveal. Indeed, w' converges to a finite limit, which is computed by setting both $w'(x)$ and $w(x)/x$ equal to the same value in (34). ■

Proof of Proposition 2. The necessity of [P] is obvious, given the characterization in Proposition 1, so we establish sufficiency.

Index each two-phase wage function $\mathbf{w}$ by its starting wage w, and define $c^*(w) = c(r, \mathbf{w})$. Condition P assures us that $c^*(w_1) \leq 1$ for some w_1. We claim that $c^*(w_2) > 1$ for some w_2. Suppose not; then $c^*(w) \leq 1$ for all w. Send $w \uparrow \infty$, then to maintain $c^*(w) \leq 1$ it must be that the associated cost-minimizing $\boldsymbol{\lambda}$ — call it $\boldsymbol{\lambda}(w)$ — converges weakly to 0. Fix any $k > 0$. Then for w large enough, [E] implies that

$$f(k, \boldsymbol{\lambda}(w))/k < r.$$

For all such w, concavity of f in k tells us that the associated cost-minimizing capital input $k(w)$ must be bounded. But now the continuity of f (together with [E]) atells us that output goes to zero as $w \to \infty$, which contradicts unit cost minimization. This proves the claim.

Because c^* is continuous,[42] there exists w^* between w_1 and w_2 such that $c^*(w^*) = 1$.

We prove that the two-phase wage function $\mathbf{w}$ emanating from w^* satisfies all the conditions for a steady state wage function. To this end, we specify a steady state wealth and bequest distribution, and occupational choice.

First, let $\boldsymbol{\lambda}^*$ be the input mix associated with the supporting wage function $\mathbf{w}$. Arrange the population over occupations according to $\boldsymbol{\lambda}^*$. Let $\theta = \theta(w^*)$.

If a family i is assigned to occupation h with $x(h) \leq \theta$, set that family's wealth equal to $\Omega(w^*, r)$, its educational bequest equal to $x(h)$, and its financial bequest equal to $[\Omega(w^*, r) - x(h)]/(1 + r)$.

Otherwise, if occupational assignment h has $x(h) > \theta$, set that family's wealth equal to $w^*(x(h))$, its educational bequest equal to $x(h)$, and its financial bequest equal to 0.

To complete the proof, we must show that each family chooses an optimal bequest. First pick a family located at occupation h with $x = x(h) \geq \theta$. Because $\mathbf{w}^*$ has a slope of at least $1 + r$ in x, this family has no need to make financial bequests. Let $M(x, x') \equiv U(w(x) - x') + V(w(x'))$ be this family's expected payoff from leaving an educational bequest x', and let $N(x) \equiv M(x, x)$. Then N is differentiable and it is easy to see that

$$N'(x) \geq U'(w(x) - x)w'(x) \text{ for all } x, \text{ with equality if } x \geq \theta. \tag{35}$$

[42] This is a consequence of the maximum theorem and the assumption that production is continuous in the weak topology over occupational distributions on $\mathcal{H}$.

For any $x' \geq x \geq \theta$, then, using the equality in (35)

$$\begin{aligned} M(x,x) &= M(x',x') - \int_x^{x'} U'(w(z)-z)w'(z)dz \\ &\geq M(x',x') - \int_x^{x'} U'(w(z)-x')w'(z)dz \\ &= M(x',x') + U(w(x)-x') - U(w(x')-x') \\ &= M(x,x'). \end{aligned}$$

Similarly, for $x' \leq x$, using the inequality in (35),

$$\begin{aligned} M(x,x) &= M(x',x') + \int_{x'}^{x} N'(z)dz \\ &\geq M(x',x') + \int_{x'}^{x} U'(w(z)-z)w'(z)dz \\ &\geq M(x',x') + \int_{x'}^{x} U'(w(z)-x')w'(z)dz \\ &= M(x',x') + U(w(x)-x') - U(w(x')-x') \\ &= M(x,x'). \end{aligned}$$

Therefore $M(x,x) \geq M(x,x')$ for all x', so that the family at x behaves optimally by bequeathing x.

Now consider a family located at $x \in [0,\theta]$. We know that its total wealth equals $\Omega(w^*,r) = w(\theta) \leq w(x)$ for all $x \geq \theta$, so by a standard single-crossing argument, and the observations of the previous paragraph, that family will never bequeath more than θ. Therefore this family must behave just as in a Becker-Tomes benchmark world with prices (w^*,r), so its assigned bequest in optimal. ■

Proof of Proposition 3: Suppose, on the contrary, that there are two steady state wage functions (modulo equivalent representations). Denote these by $\mathbf{w}_1$ and $\mathbf{w}_2$, and observe from Proposition 1 that each of them must belong to the two-phase family. Let w_1 and w_2 be the two baseline wages, and let $\theta_i = \theta(w_i)$, for $i = 1,2$. Without loss of generality suppose that $\theta_2 \geq \theta_1$.

These two wage functions *must* cross, otherwise if the profit-maximization (support) condition is satisfied at one of them it will not be satisfied at the other. Beyond θ_2 both wage functions satisfy the same differential equation (7), which rules out a crossing in this region. The functions also cannot cross below θ_1 since both wage functions are r-linear in this region. So $\theta_1 < \theta_2$, and the functions cross at some $\bar{x} \in (\theta_1,\theta_2)$.

Now proceed as in the proof of Lemma 14. Pick some x in $(\bar{x},\theta_2)$, inhabited under $\mathbf{w}_2$; then pick $x' \in (\bar{x},x)$ such that $w_1(x') = w_2(x) > 0$.

By construction, $\Omega(w_2,r) > w_2(x)$ (x is on the r-linear segment of $\mathbf{w}_2$), and so $\tilde{W}(w_2(x);w_2,r) > w_2(x)$. By Lemma 1,

$$(1+r)V'(w_2(x)) > U'(w_2(x)-x).$$

Using $w_1(x') = w_2(x)$, $x' < x$ and the concavity of U, we must conclude that

$$(1+r)V'(w_1(x')) > U'(w_1(x') - x'),$$

but this contradicts the fact that $\mathbf{w}_1$ satisfies (7) at x'. ■

Proof of Observation 4. Condition P tells us that for some two-phase wage function $\mathbf{w}$, $c(r, \mathbf{w}) \leq 1$. Define a new wage function $\hat{\mathbf{w}}$ that is r-linear from the same baseline wage as that for $\mathbf{w}$; then by Lemma 11, $\hat{w}(x) \leq w(x)$ for all x. It follows that $c(r, \hat{\mathbf{w}}) \leq 1$ as well. The existence of the required baseline wage a now follows from the same argument used in the proof of Proposition 2. ■

Proof of Proposition 4. First assume that (9) fails; using the same technique as in the proof of Proposition 2, it is easy to see that the r-linear wage function starting at a is an equal steady state. Given Proposition 3, this completes the proof.

Indeed, by the characterization result of proposition 1, an equal steady state wage function *must* be the wage function that starts at a. If, therefore, (9) holds, that proposition assures us that an equal steady state cannot exist. ■