A Theory of Return-Seeking Firms

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Abstract

We introduce a theory of return-seeking firms to study the differences between this and profit-maximising models. A return-seeking objective takes into account the opportunity cost of each additional resource input to a firm's production as being a potential capital input choice in an alternative project. We find that firm supply curves cease to exist in perfectly competitive markets, supply curves in general may slope up as well as down, that economies of scale are necessary for production, and that firms always produce on a decreasing portion of their cost curve.

1 Introduction

Standard economic models of firm behaviour assume firms maximise profits, taken to be sales revenue minus production costs, when choosing their combinations of inputs and outputs. However, in the financial management and business literature it is taken as somewhat axiomatic that the decisions of the firm are characterised by maximisation of the risk-adjusted rate of return on costs, empirically[1], and theoretically[2].

The present work seeks to understand firm behaviour if indeed their objective is to maximise the rate of return rather than profits. It is a similar exercise to that of Baumol (1959) who sought to understand firm behaviour with a maximisation of sales revenue objective. We begin by considering what rationale might be given for the rate of return on all costs to be the objective function by setting it in relation to Baumol's objective function, finding that the rationale hinges on the a firm's decision location in time in relation to their costs and revenues. This objective function is not a radical deviation from standard profit maximising, yet the outcomes for firm behaviour and their choice of production plans are significantly different under an otherwise standard model.

We find that firm behaviour in this model differs in several informative and empirically consistent ways. First, there are no restrictions on the shape of

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[1] Graham & Harvey (2001) find that 75% of firms surveyed used net present value, or, somewhat equivalently, internal rate of return decision rules.

[2] See any major text on financial management such as Berk & DeMarzo (2013, pp.207-214).

firm supply curves. In particular, price-taking firms do not respond to price changes by varying output. Second, we find the form of cost and demand schedules generate restrictions on the viable project choices that go beyond the shut down rule. We indicate how this may contribute to the understanding of firms continuing to abstain from investment (by maintaining a high hurdle) despite low or zero interest rates on capital borrowing. Third, economies of scale are necessary for the existence of production projects. Intuitively, a firm facing upwards sloping costs schedule can always lower costs by splitting its operations into smaller ones, a point we later elaborate. Fourth, we demonstrate that a firm will always operate at a point where it is exploiting economies of scale and producing at a point where costs are not increasing. We use the name return-seeking as it makes clear that while maximisation is the objective, it will not always perfectly be met at a firm, or aggregate, level, and is merely what is sought. All proofs are in the Appendix.

2 The objective function

Profit maximising through output choice assumes simultaneous incursion of costs and revenues. This an extremely restrictive assumption. Baumol's (1959) analysis considered the situation where all costs had been incurred but not revenues. To maximise return, sales revenues are maximised. Conversely, if revenues have already been incurred, maximising returns is accomplished by minimising costs. Now consider the decision about production prior to incurring either costs and revenues. Unless there is some restriction on project choice, such that there are no other opportunities for investment, returns are maximised by allocating investments so as to maximise the rate of return on each. To allocate investment beyond the point of return maximisation is to incur an opportunity cost of investment in alternative projects.

Prior to incurring any costs or revenues, all costs to be incurred are investments in obtaining a revenue stream. Therefore to maximise the rate of return is to maximise

$$R(q) = \frac{p\,(q)\,q - c\,(q)}{c(q)}.$$

where the denominator represents all costs of investment as a function of output, q, and $p(q)$ is a firm-specific inverse demand curve where all variables but own output are suppressed.

This is actually not dissimilar to profit maximisation. Maximising profits is to maximise

$$\pi(q) = p\,(q)\,q - c\,(q)$$

which is equivalent to maximising

$$\frac{\pi(q)}{\bar{K}} = \frac{p\,(q)\,q - c\,(q)}{\bar{K}}.$$

in the short run where $\bar{K}$ is some fixed level of investment, such as in fixed capital. Our objective function simply recognises that the denominator may vary as all costs are investments prior to being incurred.[3]

Out of a vector of potential production projects $\{q\} = \{q_i\}_{i=1}^{N}$, our theory of firm behaviour is that each project maximises its rate of return, that is

$$q_i = \text{argmax } R(q_i) = \text{argmax } \frac{p\left(q_i\right) q_i - c\left(q_i\right)}{c(q_i)}$$

3 Firm behaviour

For each project in our vector of potentials the return-maximising first order necessary condition is

$$c\left(q\right)\left[p(q) + q\frac{\partial}{\partial q}p(q)\right] = p(q)q\frac{\partial}{\partial q}c(q). \tag{1}$$

It is advantageous at this point to define average total costs as $ATC\left(q\right) = \frac{c(q)}{q}$ and marginal costs as $MC\left(q\right) = \frac{d}{dq}c\left(q\right)$, so that, dividing each side of Equation 1 by q we obtain a simple condition which the choice of return-maximising firms, q^*, must satisfy:

$$q^* = q : ATC\left(q\right)\left(q\frac{d}{dq}p\left(q\right) + p\left(q\right)\right) = MC\left(q\right)p\left(q\right) \tag{2}$$

Note that revenue is $p\left(q\right)q$, and we can define marginal revenue as $MR\left(q\right) = q\frac{d}{dq}p\left(q\right) + p\left(q\right)$, so Equation 2 becomes[4]

$$q^* = q : ATC\left(q\right)MR_v\left(q\right) = MC\left(q\right)p\left(q\right). \tag{3}$$

Alternatively, as function of elasticity of firm demand, ε, the condition is expressed as

$$ATC(q)[1 + \frac{1}{\varepsilon}] = MC(q). \tag{4}$$

Satisfying this condition is a maximiser if the below second order conditions are met, and the return function concave.

Proposition 1. *Suppose that $p(q)$ is demand for a normal good and is convex (demand becomes more sensitive as quantity increases) and the cost function is convex and non-upward-sloping, the production plan q^* which satisfies the first order conditions constitutes a maximiser of returns if and only if*

$$-\{p'(q)[2c(q) - qc'(q)] + \frac{q}{c(q)}p(q)c'(q)\}$$
$$\geq qc(q)[p''(q) - p'(q)] + p(q)[qc''(q) - 2c'(q)] \geq 0$$

[3]A consistent empirical finding is that *all* resource inputs vary with output for most firms. Miller (2001) explains that "for any given plant, factor usage differs as (the rate of) output is altered. To reduce output, labor is laid off. So too is the plant. The flows of all factor services (not just labor services and raw materials) change when output is increased or reduced." Not only does capital utilisation vary with output, but surveys also find that the textbook variable cost of labour has characteristics of fixed capital due to contractual arrangements (Oi, 1962; Kambourov & Manovskii, 2009), which adds to that case that the theoretical distinction between types of inputs and their costs in the standard model does not have a strong empirical basis.

[4]Notice also that this equation nests the special case of $MC(q) = MR(q)$ when $ATC(q) = p(q)$ and $ATC'(q) = p'(q)$.

Notice that Proposition 1 implies there may be certain projects with exotic properties that will never be undertaken as they do not have a return maximal output. These properties are over and above the shutdown condition of profit maximisation, as it is not merely that price does not exceed average variable costs which leads to a project being excluded from the vector of projects undertaken. The main lesson we take from this condition is that in order to be engaged in production a project must have a particular mix of cost and demand schedules with particular responses and rates of responses to output.[5] There may be no feasible price and cost combination to ensure investment occurs, which may be important to understanding the response (or lack thereof) of investment to interest rates or wages (Sharpe & Suarez, 2013; Lane & Rosewall, 2015).

4 The price-taker

In the special case price-taking, it follows that the price the firm receives for its output is independent of its output, such that

$$\frac{d}{dq}p\left(q\right)=0\implies p\left(q\right)=p. \tag{5}$$

If this is indeed the case then applying Equation 5 to the fundamental condition (Equation 2) on the choice of a return-maximising firm we find that the production plan under a competitive market, q_c^* must satisfy a special case of Equation 2 where $\frac{d}{dq}p\left(q\right)=0$:

$$q_c^*=q:ATC\left(q\right)=MC\left(q\right) \tag{6}$$

Hence the return-maximising firm in a competitive market seeks to produce that amount which equates its average and marginal costs, regardless of the price of their output, provided that that price is above costs. In other words, average total costs are minimised. This may seem a rather unusual statement, though when we consider that firms in a competitive market have no control over their prices, while they do over their costs, it would make sense for them to focus on costs in their decision making process unless prices are so low as to make their activities unprofitable. It is fairly straightforward to demonstrate an intuitive result following from Equation 6 that firms which have no control over the price of their output seek to minimise average total costs in order to maximise return. This is consistent with the case earlier discussed of firms who have already received revenue, and have no control over it, and hence are only able to focus on their own costs.

Proposition 2. *If $\frac{d^2}{dq^2}c\left(q\right)>0\,\forall\,q\in Q$ (that is, the cost function is globally convex), and the cost function is second-order but not third order differentiable then*

$$q_c^*=q:ATC\left(q\right)=MC\left(q\right)\implies q_c^*=argmin_{q\in Q}ATC\left(q\right).$$

[5] Of note is that if the demand and cost curves are both linear there is no solution. If the slope of the cost curve is always less than the demand curve, there is no solution.

Since q_c^* is not a function of price under price-taking, we have but a trivial supply correspondence in that the firm supply curve is vertical above the cost-minimising level of output.

5 The price-setter

In general, we can take *firm-specific* demand to be a non-specified function of the firm's output, $p(q)$. In this case, the firm's production plans must satisfy the general condition in Equation 2. We can establish that the price setter restricts output relative to the price taker.

Proposition 3. *For a return-maximising firm producing a normal good, the production plan for a non-competitive firm entails restricting supply relative to the competitive supply. That is, for* $q^* \neq 0$ *satisfying 2, and for* $\frac{d}{dq}p(q) \leq 0$

$$q_c^* \geq q^*.$$

While this result is intuitive, we will see in the next section (in the theorems it supports) that it can lead to unusual results. We can also demonstrate that, consistent with post-Keynesian models, prices are *always* a markup over marginal costs. First we establish a corollary.

Corollary 1. *The production plan of a return maximising firm for a normal good is in general such that average total costs are higher than marginal costs. That is, for* q^* *satisfying Equation 2, and for* $\frac{d}{dq}p(q) \leq 0$

$$ATC(q^*) \geq MC(q^*).$$

This allows us to establish the existence of the markup.

Proposition 4. *Any non-trivial*[6] *production plan of a return maximising firm for a normal good is in general such that prices are a mark-up over its marginal costs. That is, for* $q^* \neq 0$ *satisfying 2, and for* $\frac{d}{dq}p(q) \leq 0$

$$p(q^*) \geq MC(q^*).$$

This is as expected for a price-setter. In contrast, the main result of interest is that the correspondence between a firm's output and price may be positive or negative, implying that a firm-level supply curve can also slope downwards in response to a rightward shift in the firm-level demand curve (a result graphically demonstrated in Panel (d) of Figure 1).

Proposition 5. *Suppose there is a positive demand shock caused by some variable a. That is,* $\frac{\partial p(q)}{\partial a}\partial a > 0$*. Price will decrease,* $\partial p(q*) < 0$ *if an only if:*

$$\partial q^* > \left(-\frac{\partial p(q^*)}{\partial q^*}\right)\frac{\partial p(q^*)}{\partial a}\partial a > 0.$$

[6] Meaning any production plan which involves some form of actual production rather than the trivial no-production production plan $q^* = 0$.

Proposition 5 requires that there be large enough response to the demand shift to outweigh the increase in price. Observe that ∂q^* is governed by the dynamics of Equation 2, which we have deferred considering formally, as it would be quite involved. Most notable is the dependence of ∂q^* on average total cost. Since we established on Corollary 1 that $ATC(q) \geq MC(q)$, if we have a shift in the demand, we will see the equality in Equation 2 broken unless there is a reduction in ATC, an increase in MC, or a reduction in $p(q)$, or both. If we have economies of scale though, ATC will be reducing, led by a reducing MC, making it more likely that return maximisation will require a reduction in price.

This implication accords well with the empirical record (Walters, 1963; Blinder *et al.* , 1998; Miller, 2001), perhaps more so than profit maximising. This intuition is further explored in the Section 7.

6 Economies of scale

Another core aspect of our model that differs from the standard profit-maximisation is the central role of economies of scale, which are necessary for any production to place.

Proposition 6. *If supply is non-trivial then there must be a region where average total costs are non-increasing. That is,* $q_c^* > 0 \implies \exists\, [q_1 \quad q_2] \subset Q : \frac{d}{dq} ATC\,(q) \leq 0$

The intuition behind this result is that a firm facing strictly increasing costs can simply dissolve into smaller firms to increase returns, since in our model firms themselves arise from choices to engage in the production of a certain amount of output. Such a process can continue until each new smaller firm converges to a production level of zero, or until the new firms begin to face economies of scale in their production. We believe this result sheds some light on questions surrounding the existence of firms and firm size, in that a firm will only exist if expanding production can reduce its unit cost, and empirical findings that most firms produce at point where they face economies of scale (Blinder *et al.* , 1998; Miller, 2001).

We can further demonstrate that a return-maximising firm will always produce at a point where economies of scale are being realised, meaning at a point where average total costs are decreasing in output.

Proposition 7. *For a return-maximising firm producing a normal good with a convex cost function, the production plan for any firm will always exist on a downward-sloping portion o the supply curve, that is* $\frac{d}{dq} c\,(q) \leq 0\ \forall\ q = q^*$.

7 Discussion

With the aid of Figure 1 we illustrate the core elements of our theory. Panel (a) of Figure 1 illustrates return-seeking optimal firm output q^* in the case of price-taking.[7] Some descriptive points are of note. First, return maximisation corresponds with cost minimisation as per Proposition 2, implying that firm

[7] q^* represents return-maximising output, while q_p shows profit-maximising output.

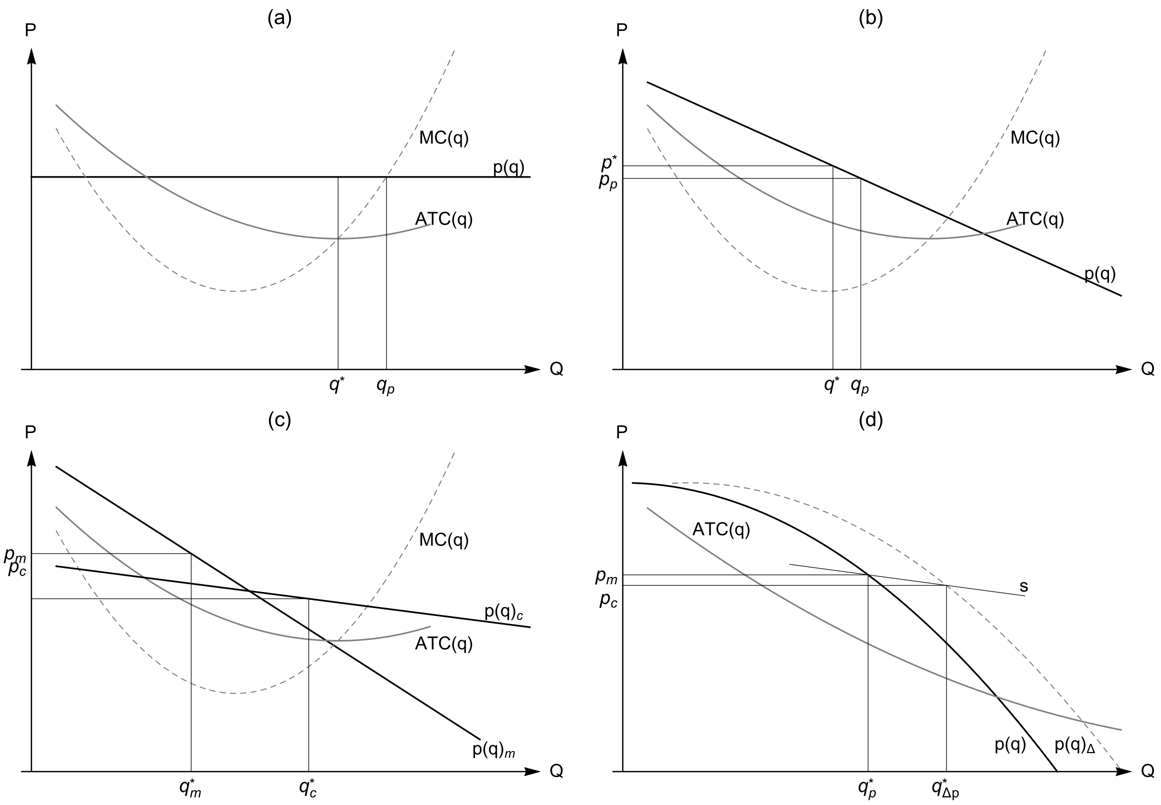

Figure 1: Illustrations of optimal output for return-seeking firms

output will be unresponsive to any shift in the demand curve. Second, without additional assumptions on the position of the firm's demand curve, there is no implication that firm output is socially optimal in terms of prices accurately reflecting production costs. Third, a return-seeking firm has less output than an analogous profit-maximising firm, and hence we can also intuit that it will underutilise capital in comparison.

The general case of price-setting is illustrated in Panel (b). We see graphically that $ATC(q^*)$ must always be non-increasing up to, and including, the limiting case of a competitive price-taking firm in Panel (a). As per Proposition 3, optimal output of a return-seeking firm, q^* is less than that of the profit maximiser, q_p.

Panel (c) allows us to show the relationship between the slope of firm demand and its relative focus on cost minimisation. Compared to the cost-minimising amount we see the firm restricting output in response to its degree of price-setting power. As we demonstrated, price-setting power decreases (as demand flattens from the more monopolistic $p(q)_m$ to the more competitive $p(q)_c$), the return-seeking firm shifts its focus from restricting output and increasing prices to minimising costs which aligns with a body of evidence showing that firm mark-up over costs are a function of pricing power (Kalecki, 1952; Sraffa, 1926). This result is consistent with profit maximisation models whereby competition reduces prices (see Lerner (1934)) and increases output. To be clear, in our model competitiveness need not be solely dependent on the number of firms in a market for an homogenous good since $p(q)$ is a firm-specific demand curve rather than a market demand curve. The firm-specific demand curve can be determined by any number of factors, for example consumer access to substitutes generally.[8]

Return-maximising output increases in response to a positive shift in the demand curve, which is illustrated by way of the response of q^* in Panel (d). In Panel (d) we also observe that it is not necessarily the case that an increase in demand will result in a price increase, even when a firm's cost curve is fixed. Indeed, depending on the relative slopes of the demand and cost curves, shifts in demand may generate downward-sloping supply curves for an individual firm, and perhaps for a market broadly defined if its member firms face similar cost structures. This result is expected when firms facing relatively steep demand curves, but extensive economies of scale yet to be utilised, which is consistent with the empirical estimates (Shea, 1993). Supply can thus be broadly seen to respond positively to demand increases, but the supply curve, defined as the relationship between output supplied and price, may actually be upward or downward sloping.

8 Conclusions

In this paper we have built a theory of firm behaviour by departing from the strict assumption of profit maximising that requires costs and revenues to be simultaneously incurred, and considering the case where neither costs nor revenues have been incurred. In this situation each additional cost for a firm is a potential investment, and hence carries the opportunity cost of being an investment in an alternative project. While this assumption alters slightly the objective function of the theory compared to profit-maximisation, the implications that follow are radically different.

In the limiting case of price-taking we no longer have a firm-level supply curve in any meaningful sense, as firms supply at their cost minimising point and will hence only expand supply if they are able to shift this point of their cost curve to the right. In the general case of price-setting, there are no restrictions on the shape of firm supply curves. For return-seeking firms there are restrictions on the viable project choices that go beyond the shut down rule due to the necessary form of the cost and demand schedules.

A more interesting and implication of this model is that economies of scale are necessary firm production to exist.

While firms in an imperfectly competitive market price above marginal cost and thus are pricing inefficiently by restricting supply relative to its competitive level, they at least respond to demand to satiate it (under certain conditions), and in this sense imperfectly competitive markets become a *more* efficient market structure than perfectly competitive markets. Firm supply curves may also take any shape for firms operating in imperfect markets.

[8] This model, we emphasise, is at a firm level. Any aggregation method to study market-wide or macro-economic phenomena must account for minor deviations from perfect expectations at the micro level (hence our label of a return-seeking theory rather than of return-maximising) which may interact to produce major deviations at a macro level. Such micro level deviations need not be independent and randomly distributed but may systematically deviate at orders of magnitude below the resulting macro-economic deviations.

This model may negate many results of profit maximisation that have over the years become canon and perhaps viewed as indispensable, but in doing so it replaces these results with interesting new ones and opens up new lines of theoretical and empirical investigation to flesh out all the intricate subtleties of the return-seeking firm.

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Appendix: Proofs

Proof of Proposition 1: To establish any such turning point is a maximum we require $R''(q) \leq 0$ (i.e. concavity)

$$R''(q) = \frac{d}{dq}\left[\frac{c(q)[p^{'}(q)q + p(q)] - p(q)qc^{'}(q)}{c(q)^2}\right]$$

$$= \frac{d}{dq}\left[\frac{p^{'}(q)q + p(q)}{c(q)} - \frac{p(q)qc^{'}(q)}{c(q)^2}\right].$$

Expanding.

$$R''(q) = \frac{c(q)[p''(q)q + p'(q)] - p'(q)qc(q) + c(q)p'(q) - p(q)c'(q)}{c(q)^2}$$

$$- \frac{c(q)^2\{c'(q)[p'(q)q + p(q)] + c''(q)p(q)q\} - p(q)qc'(q)c(q)}{c(q)^4}.$$

Now we engage in a series of manipulations of

$$R''(q) \leq 0 \Leftrightarrow \frac{c(q)[p''(q)q + p'(q)] - p'(q)qc(q) + c(q)p'(q) - p(q)c'(q)}{c(q)^2}$$

$$\leq \frac{c(q)^2\{c'(q)[p'(q)q + p(q)] + c''(q)p(q)q\} - p(q)qc'(q)c(q)}{c(q)^4}.$$

By which we arrive at the important condition

$$R''(q) \leq 0 \Leftrightarrow c(q)^2\{qc(q)\{p''(q) - p'(q)\} + p(q)[qc''(q) - 2c'(q)] + p'(q)[2c(q) - qc'(q)]\}$$

$$+p(q)qc'(q)c(q) \leq 0.$$

We assume typically that we have a normal good ($p'(q) \leq 0$), and we proved in the paper that return-seeking firms always produce at a point where costs are decreasing unless they are constrained not to do so ($c'(q) \leq 0$), and have a convex cost function (so $c'(q) \geq 0$). We always have positive costs and positive prices and quantities (so $p(q) \geq 0$ and $c(q) \geq 0$ and $q \geq 0$). If we have a convex demand function ($p''(q) \leq 0$) this would suggest that as quantity increases demand become more sensitive to price, which would make sense if our preferences were convex as more quantity *ceteris paribus* would satisfy less and less. Now with these assumption in hand, we can establish that

1. $q \geq 0, c(q) \geq 0, p''(q) > 0, p'(q) \leq 0$ ($\Rightarrow -p'(q) > 0$)
   $\Rightarrow qc(q)\{p''(q) - p'(q)\} \geq 0$

2. $p(q) \geq 0, q \geq 0, c''(q) \geq 0, c'(q) \geq 0$ ($\Rightarrow -c'(q) > 0$)
   $\Rightarrow p(q)[qc''(q) - 2c'(q)] \geq 0$

3. $p'(q) \geq 0$ ($\Rightarrow -p'(q) > 0$)$, q \geq 0, c(q) \geq 0, c'(q) \geq 0$ ($\Rightarrow -c'(q) > 0$)
   $\Rightarrow -p'(q)[2c(q) - qc'(q)] \leq 0$

4. $p(q) \geq 0, c'(q) \geq 0, q \geq 0, c(q) \geq 0$
   $\Rightarrow p(q)qc'(q)c(q) \leq 0$

Since in any case $c(q)^2 \geq 0$ we can determine using arguments 1 through 4 that

$$c(q)^2\{qc(q)\{p''(q) - p'(q)\} + p(q)[qc''(q) - 2c'(q)]\} \geq 0$$

and

$$c(q)^2 p'(q)[2c(q) - qc'(q)] + p(q)qc'(q)c(q) \leq 0$$

and so $R''(q) \leq 0$ if and only if

$$-\{c(q)^2 p'(q)[2c(q) - qc'(q)] + p(q)qc'(q)c(q)\}$$

$$\geq c(q)^2\{qc(q)\{p''(q) - p'(q)\} + p(q)[qc''(Q) - 2c'(q)]t\} \leq 0.$$

Dividing through be $c(q)^2$ and rearranging for a little more nicety of expression we arrive at the result.

QED

Proof of Proposition 2: The first order necessary conditions for a minimisation problem require that

$$\frac{d}{dq} ATC(q) = \frac{d}{dq} \frac{c(q)}{q} = 0$$

hence the first order necessary condition for the average total cost minimisation is

$$\frac{q \frac{d}{dq} c(q) - c(q)}{q^2} = 0 \implies ATC(q) = MC(q).$$

Now, for a second-order but not third order differentiable average total cost function, the Taylor expansion about q is (for some constant κ)

$$ATC(q') = ATC(q) + (q' - q) \frac{d}{dq} ATC(q) + (q' - q)^2 \frac{1}{2} \frac{d^2}{dq^2} ATC(\kappa)$$

the first order conditions imply that

$$ATC(q') = ATC(q) + (q' - q)^2 \frac{1}{2} \frac{d^2}{dq^2} ATC(\kappa).$$

So q minimises $ATC(q)$ if and only if $ATC(q) < ATC(q')\ \forall q' \neq q$, which would require that $\frac{d^2}{dq^2} ATC(\kappa) > 0$ and the average cost function be convex, which is indeed the case by assumption. Hence since q_c^* must satisfy the first order necessary conditions, it is indeed a minimiser of average total cost.

QED

Proof of Proposition 3: From Proposition 2, we have that $q_c^* = \text{argmin}_{q \in Q} c(q)$ since $ATC(q_c^*) = MC(q_c^*)$. But in non-competitive cases q^* does not satisfy this, by the fundamental condition on the choice of a return-maximising firm (Equation 2), and hence $ATC(q^*) \geq ATC(q_c^*)$.

Suppose by way of contradiction then that $q^* > q_c^*$. Then as above we have $ATC(q^*) \neq \min_{q \in Q} ATC(q)$. However, in almost all cases q is a normal good, so $\frac{d}{dq} p(q) \leq 0$, so $p(q^*) < p(q_c^*)$. Hence $q^* > q_c^*$ contradicts the definition of $q^* = \text{argmax}_{q \in Q} R(q)$, since prices can be increased and costs decreased, and return thus increased by changing production plans from q^* to q_c^*. Hence it must be the case that $q^* \leq q_c^*$.

QED

Proof of Corollary 1: This proposition immediately follows from the definition of q^* as satisfying condition 2, which may be rewritten as

$$\frac{\left(q^* \frac{d}{dq} p(q^*) + p(q^*)\right)}{p(q^*)} = \frac{MC(q^*)}{ATC(q^*)}$$

Now, if the output is a normal good the demand curve has a negative slope, so we have

$$\frac{d}{dq} p(q^*) \leq 0 \implies \frac{\left(q^* \frac{d}{dq} p(q^*) + p(q^*)\right)}{p(q^*)} \leq 1 \implies \frac{MC(q^*)}{ATC(q^*)} \leq 1$$

And thus it must be the case for 2 to hold that $ATC(q^*) \geq MC(q^*)$.

QED

Proof of Proposition 4: The same conditions here are assumed as for Corollary 1, and so it must be the case that $ATC(q^*) \geq MC(q^*)$. Now if we can show that $ATC(q^*) \leq p(q^*)$, then by the transitivity of the real numbers $\mathbb{R}$ upon which all these variables are defined, we will have the result.

Now, that $ATC(q^*) \leq p(q^*)$ is in fact implied by the definition of q^* which states that

$$q^* = \text{argmax}_{q \in Q} R(q) = \text{argmax}_{q \in Q} \left\{ \frac{p(q)\, q - c(q)}{c(q)} \right\}.$$

To see this, suppose by way of contradiction that $ATC(q^*) > p(q^*)$ and $q^* \neq 0$. Then $p(q^*)\, q^* < c(q^*)$ and $R(q^*) < 0 \, \forall \, q \in Q \backslash \{0\}$. But this contradicts q^* being a maximiser of returns, since $R(0) = 0 > R(q^*)$. Hence in all non-trivial production plans, we must have $ATC(q^*) \leq p(q^*)$, while we have from above Corollary 1 that $ATC(q^*) \geq MC(q^*)$. So by the transitivity of the real numbers $\mathbb{R}$ we can state that

$$p(q^*) \geq ATC(q^*) \geq MC(q^*) \implies p(q^*) \geq MC(q^*).$$

QED

Proof of Proposition 5: We have $\frac{\partial p(q)}{\partial a} > 0$ by assumption. But then also $\frac{\partial q^*}{\partial p} > 0$, by definition. So

$$\begin{aligned} \partial p(q^*) &\Leftrightarrow \frac{\partial p(q^*)}{\partial a}\partial a + \frac{\partial p(q^*)}{\partial q^*}\partial q^* < 0 \\ &\Leftrightarrow \partial q^* \frac{\partial p(q^*)}{\partial q^*} < \frac{\partial p(q^*)}{\partial a}\partial a. \end{aligned}$$

$\frac{\partial p(q^*)}{\partial q^*}$ is simply slope. The variable of interest is ∂q^*

$$\partial p(q^*) < 0 \Leftrightarrow \partial q^* > \left(-\frac{\partial p(q^*)}{\partial q^*}\right)\frac{\partial p(q^*)}{\partial a}\partial a > 0$$

since $\frac{\partial p(q^*)}{\partial a} > 0$, $\partial a > 0$ and $\frac{\partial p(q^*)}{\partial q^*} < 0$.

QED

Proof of Proposition 6: Suppose that q_c^* is indeed a minimiser of costs. Hence it must be the case about q_c^* that $ATC\left(q_c^* - \varepsilon\right) > ATC\left(q_c^*\right)$ for some range of $\varepsilon > 0$. Let us suppose, by way of contradiction, that $\nexists\,[q_1 \quad q_2] \subset Q : \frac{d}{dq}ATC\left(q\right) \leq 0$. This then implies that average total costs must be everywhere increasing in q, that is, $\frac{d}{dq}ATC\left(q\right) > 0\,\forall\, q \in Q$. But if this then were the case, we would have $ATC\left(q_c^* - \varepsilon\right) < ATC\left(q_c^*\right)$, which contradicts that $q_c^* > 0$ is a minimiser of average total costs, because only for $q_c^* = 0$ is $ATC\left(q_c^* - \varepsilon\right) \nless ATC\left(q_c^*\right)$, and even then this is only true trivially as $-\varepsilon \notin Q$. Hence it must be the case that for a minimiser to exist there must be some region of Q for which average total costs are decreasing (strictly speaking, non-increasing).

QED

Proof of Proposition 7: From Proposition 2, we have that $q_c^* = q : \frac{q}{dq}c(q) \leq 0$. If the cost function is convex then $\frac{d^2}{d^2q}c(q) \geq 0$, so $\frac{d}{dq}c(q_1) \geq \frac{d}{dq}c(q_2) \forall q_1 \geq q_2$. From Proposition 3 we have that $q_c^* \geq q*$ and it follows that $\frac{d}{dq}c(q_c^*) \geq \frac{d}{dq}c(q^*)$, and since $\frac{d}{dq}c(q_c^*) = 0$ it must be the case that $\frac{d}{dq}c(q^*) \leq 0 \forall q^*$.

QED