Explaining the paradox of Allais: a hyper-Bayesian account

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Abstract: In this paper we show that decision makers who maximize expected utility on a two-dimensional set of the real numbers (i.e., who maximize expected utility on $\mathbb{R} \times \mathbb{R}$) rather than on the real numbers alone, can act in ways predicted by the Allais paradox. In particular, we show that the 'common consequence effect' and the 'common ratio effect' as described by Allais in his original 1953 paper can be explained by the extended expected utility model that is described in this paper. The main part of the argument is presented in cognitive-functional terms, whilst a behavioural-axiomatic foundation is provided in an appendix.

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It's recently been shown that a model in which decision makers maximize expected utility on a ring of hypercomplex numbers can be used to explain the paradoxes of Ellsberg and Machina (Sharpe, 2015). In this paper we show that the same model of 'hyper-Bayesian' behaviour can also account for the paradox of Allais.

In part I of the paper, we describe the ring of hypercomplex numbers, and point out that it's isomorphic to the two-dimensional direct sum (or, equivalently, the direct product) of the real numbers (i.e., we point out that the ring is structurally identical to $\mathbb{R} \times \mathbb{R}$). Additionally, we describe how decision makers use those numbers in their decision making – in particular, we show how decision makers extend the expected utility principle to the hypercomplex domain.

In part II of the paper, we show that the given model of decision making can indeed account for the common consequence effect and the common ratio effect, both of which are instantiations of Allais' paradox. We use the examples that Allais provided in his original (1953) paper as our cases.

The results are discussed in part III of the paper. In the Appendix, we provide axiomatic foundations for the model described in part I.

Part I

The ring, $\acute{E}$, on which decision makers operate is composed of elements that have the form: $a + b\acute{e}$, where $a, b \in \mathbb{R}$, and $\acute{e}$ is an operator on elements. The first part of the element, $a + b\acute{e}$, is referred to as the real part, and the second as the é-real part. Speaking informally, we say that a is the 'exact' or the 'unambigusous' part of the number (and hence is untagged), whilst b is the 'inexact' or 'ambiguous' part of the number (and consequently is tagged by $\acute{e}$).

Addition (+) and multiplication (· or juxtaposition) on $\acute{E}$ are defined as follows:

$$(a + b\acute{e}) + (c + d\acute{e}) = (a + c) + (b + d)\acute{e}$$

$$(a + b\acute{e}) \cdot (c + d\acute{e}) = (ac) + (ad + bc + bd)\acute{e}$$

The addition operation is, perhaps, obvious (it's the same as the addition operation for the complex numbers). The multiplication operation is justified on the grounds that the product of any two 'exact' or 'unambiguous' numbers is also an exact or unambiguous number (hence, ac is an unambiguous number since both a and c are unambiguous); whilst any time an 'inexact' or an 'ambiguous' number multiplies any other kind of number, the result is an inexact or an ambiguous number (hence, $ad + bc + bd$ is an ambiguous number since both b and d are ambiguous, and it's tagged by the operator $\acute{e}$ to indicate that fact).

This algebra describes a semi-local ring that's isomorphic to the split complex numbers, and hence is also isomorphic to the direct sum: $\mathbb{R} \oplus \mathbb{R}$. (For further discussion of the ring, see Izhakian & Izhakian, 2014, who introduced it to decision theory; also, see Sharpe, 2015).

In addition to the above operations, we can define the 'right-angle' value of each element of $\acute{E}$ as follows:

$$\lfloor a + b\acute{e} \rfloor = a + b \in \mathbb{R}$$

The existence of right angle values gives rise to right angle addition since:

$$\lfloor a + b\acute{e} \rfloor + \lfloor c + d\acute{e} \rfloor = (a + b) + (c + d) = \lfloor (a + c) + (b + d)\acute{e} \rfloor = \lfloor (a + b\acute{e}) + (c + d\acute{e}) \rfloor$$

The existence of right angle values and right angle addition is useful in characterizing the epistemic evaluations that decision makers make.

The epistemic evaluations – or beliefs – of decision makers are structured as follows: given a sample space, Ω, and an associated algebra, 2^{Ω} (Ω, finite), and given that $A, B, C, \ldots \in 2^{\Omega}$ denote events, there's a composite mapping: $\mu_{\acute{e}} = \hat{\mu}_{\acute{e}} \circ \lfloor \mu_{\acute{e}} \rfloor$, with $\lfloor \mu_{\acute{e}} \rfloor : 2^{\Omega} \to \mathbb{R}$ and $\hat{\mu}_{\acute{e}} : \mathbb{R} \to \acute{E}$, with $\lfloor \mu_{\acute{e}} \rfloor$ satisfying: $0 \leq \lfloor \mu_{\acute{e}}(A) \rfloor \leq 1$; $\lfloor \mu_{\acute{e}}(A \cup B) \rfloor = \lfloor \mu_{\acute{e}}(A) \rfloor + \lfloor \mu_{\acute{e}}(B) \rfloor$ when $A \cap B = \emptyset$; and $\mu_{\acute{e}}(\Omega) = 1$ (which implies that: $\lfloor \mu_{\acute{e}}(\Omega) \rfloor = 1$); whilst $\hat{\mu}_{\acute{e}}$ ensures that: $\lfloor \mu_{\acute{e}}(A) \rfloor = \lfloor a + b\acute{e} \rfloor$ and $a, b \geq 0$. Thus, $\mu_{\acute{e}}(A)$, can be thought of as the composition of a probability measure defining the right angle values of beliefs, $\lfloor \mu_{\acute{e}}(A) \rfloor$, along with a decomposition of those beliefs into real and é-real parts (or components), $a + b\acute{e}$.

When the decision problem with which decisions makers are confronted specifies 'objective' probabilities – as is the case in Allais-type decision problems – then the individual's subjective probability mapping, $\lfloor \mu_{\acute{e}} \rfloor$, coincides with that objective probability, which is denoted by π. This is to say, given the objective probability of some event, A, from the set of events, Ω, we have: $\pi(A) = \lfloor \mu_{\acute{e}}(A) \rfloor$.

With regards to the decomposition of probabilities into the two parts of the é-real numbers, decision makers are assumed to assign beliefs that they regard as 'unambiguous' or 'secure' or 'not susceptible to fudging' to the pure real numbers, whilst beliefs that are 'ambiguous', or 'less secure', or are 'more prone to fudging', are assigned to the pure é-real numbers. In this way, the é-real numbers allow decision makers to express their attitudes towards 'ambiguity' and 'vagueness' in ways that the real numbers can't do on their own. These beliefs are then used in the decision making process described immediately below. The process is described here in its functional form – an axiomatic foundation for the implied behaviour is provided in the Appendix.

Assumption 1: The choice domain is the set of lotteries. The choices confronting decision makers are 'lotteries', which specify an é-real probability for each possible 'outcome' or 'consequence' or 'prize'. Prizes are denoted by x, with $x \in X$, where X is the set of all possible prizes (and: $\#X < \infty$). Hence, each lottery, L, is defined as:

$L = (\mu_{é}^{L}: X \rightarrow \{a + bé \mid \lfloor a + bé \rfloor \leq 1; 0 \leq a, b; \sum_{x \in X} \lfloor \mu_{é}^{L}(x) \rfloor = 1\})$

Elementary events in the sample space are identified by the consequences they bring about. Hence, we can say that whenever the given decision problem specifies the probability of some prize, $x \in X$, being attained – that is, when the decision problem specifies: $\pi(x)$ – then we have: $\pi(x) = \lfloor \mu_{é}(x) \rfloor$.

The set of lotteries is denoted by $\mathcal{L}$.

Assumption 2: Computation of expected utility on $\acute{E}$. Decision makers compute a particular form of expected utility whose value lies on the ring:

$$v = \sum_{x \in X} \mu_{é}(x) \cdot u(x)$$

Where: $u(x)$ is the utility of a consequence; and, $\mu_{é}(x)$ is the belief in the likelihood of that outcome occurring. The (expected) value of an act is denoted by v.

Utilities are defined on $\mathbb{R}$, whilst beliefs are given on $\acute{E}$ ($\mathbb{R}$ is a subring of $\acute{E}$). The utility function is defined over dollar values and has the usual properties. As $u(.)$ and $\mu_{é}(.)$ are well-defined, and the operations of addition and multiplication on E are also well-defined, so too is the value function, v.

Assumption 3: Consistent attitude to ambiguity. The decision maker's attitude towards ambiguity is represented by a real number: $\alpha > 0$, which is used to define the linear transformation: $\phi: \acute{E} \rightarrow \mathbb{R}$ with $a + bé \mapsto \alpha a + b$. The complete decision rule for the decision maker thus becomes:

$$\max_{L \in \mathcal{L}} \ \phi\left(\sum_{x \in X} \mu_{é}(x) \cdot u(x)\right)$$

If the decision maker is ambiguity averse, then $\alpha > 1$, and if the decision maker is ambiguity avid, then $\alpha < 1$. The intuition here is that, if $\alpha > 1$, the decision maker needs a greater increase in the é-real value of an act to compensate for a given loss in the real value of that act if she is to remain indifferent between the two acts. If $\alpha < 1$ the decision maker needs less than proportionate compensation for taking on greater ambiguity. If ambiguity has no impact on the decision maker, we have: $\alpha = 1$. Since the linear transformation, ϕ, is applied to all expected values on the vector space defined by the é-real numbers, it gives rise to an equivalence relation that can be represented by an indifference map as in Figure 1 (which represents a situation of ambiguity aversion since $\alpha > 1$).

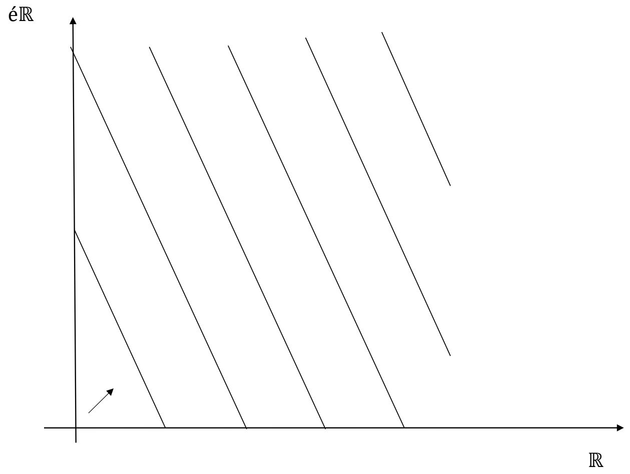

Figure 1

Part II

Case 1: the Allais paradox – the common consequence effect

Here is the Allais paradox in tabular form:

	A: 1%	B: 89%	C: 10%
f	\$1,000,000	\$1,000,000	\$1,000,000
g	\$0	\$5,000,000	\$1,000,000
f'	\$1,000,000	\$1,000,000	\$0
g'	\$0	\$5,000,000	\$0

Canonical decision theory requires that, if lottery f is chosen in preference to lottery g, then f' must be chosen over g', and vice versa. It's well known that this is often honoured in the breach.

The way this behaviour is accounted for in our model runs as follows. First, let $u(\$1{,}000{,}000) = \bar{u}$, $u(\$0) = \bar{u}M$, and $u(\$5{,}000{,}000) = \bar{u}N$, with: $0 < M < 1 < N$. Next, we recall from our earlier discussion that decision makers initially assign probabilities which they then decompose into real and é-real parts (or 'unambiguous' and 'ambiguous' parts, respectively), and they use these decomposed numbers to represent the beliefs they work with in their expected value computations. The decision maker in this case behaves in just this manner, with the following representations of beliefs:

$\mu(\emptyset) = 0,\ \mu(A) = \frac{1}{100}\acute{e},\ \mu(B) = \frac{89}{100}\acute{e}, \mu(C) = \frac{10}{100},\ \mu(A \cup B) = \frac{90}{100},\ \mu(A \cup C) = \frac{11}{100}\acute{e},$ $\mu(B \cup C) = \frac{99}{100}\acute{e}$, and $\mu(A \cup B \cup C) = 1$.

We note here that these decompositions into real and é-real parts are consistent with the original probabilities of the paradox since those probabilities are preserved by the right angle values of the é-real numbers so decomposed. The ambiguous assignments reflect the fact that the decision maker treats numbers with non-zeroes defined to more than one decimal place as 'less secure' or 'more ambiguous' or more prone to 'fudging' than 'to-one-decimal-point' numbers. We note also that, in this case, the ambiguity isn't 'in nature', as it is in the Ellsberg paradox (where the ambiguity is related to the fact that probabilities aren't objectively given); rather, in the Allais case, ambiguity is the result of the decision maker actively deciding to treat objectively given probabilities in different ways according to some characteristic properties of those probabilities.

Given these belief assignments, the values of the lotteries are:

$f: \bar{u}$	$f': (.90 + .10M)\bar{u}$
$g: (.01M + .89N)\bar{u}\acute{e} + .10\bar{u}$	$g': (.11M + .89N)\bar{u}\acute{e}$

Now define: $\beta = \frac{.01M + .89N}{.90}$ and $\gamma = \frac{.11M + .89N}{.90 + .10M}$. We suppose that $\beta < 1$ on the basis that the marginal utility of the increment from \$1,000,000 to \$5,000,000 is so much less than that for the increment from \$0 to \$1,000,000. We then necessarily have: $\beta < \gamma$. If, in addition, it turns out that: $\beta < \alpha < \gamma$, we get the behaviour predicted by the paradox. In this manner, the model accounts for Allais' common consequence effect.

The paradox is depicted in Figure 2, where the dashed lines describe the two pairwise choices; and the sloped solid lines are 'indifference curves' depicting the decision maker's attitude to ambiguity, each with a slope of $-\alpha$.

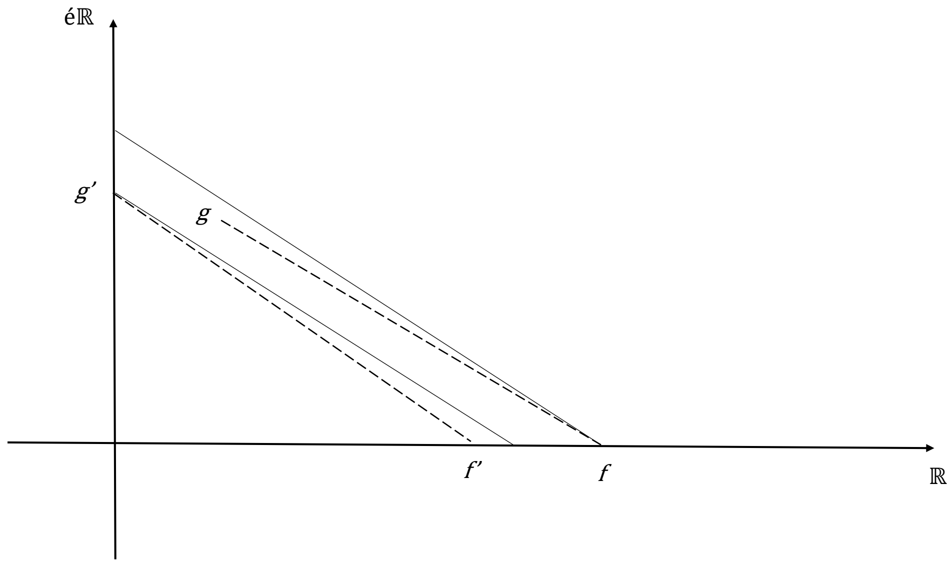

Figure 2

Case 2: the Allais paradox – the common ratio effect, per Allais (1953)

The common ratio effect is accounted for in a similar fashion to the common consequence effect.

In the given table, the top left quadrant describes lottery, f, which is to be compared to lottery, g, which is in the bottom left of the table; lotteries f' and g' in the right half of the table are to be compared.

Choice A	A: 100%	A^C: 0%	Choice C	C: 1%	C^C: 99%
f	$100m	$0	f'	$100m	$0
Choice B	B: 98%	B^C: 2%	Choice D	D: 0.98%	D^C: 99.02%
g	$500m	$0	g'	$500m	$0

In each option, there are two states with known probabilities, and canonical decision theory requires that, if f is chosen in preference to g, then f' must be chosen over g', and vice versa. Often, these preference relations fail to hold.

The way this behaviour is accounted for in our model runs as follows. First, let $u(\$100) = \bar{u}$, $u(\$0) = \bar{u}M$, and $u(\$500) = \bar{u}N$, with: $0 < M < 1 < N$. Next, the decision maker determines her beliefs:

$\mu(A) = 1,\ \mu(A^C) = 0,\ \mu(B) = \frac{98}{100},\ \mu(B^C) = \frac{2}{100},\ \mu(C) = \frac{1}{100},\ \mu(C^C) = \frac{99}{100},\ \mu(D) = \frac{98}{10000}é,$ $\mu(D^C) = \frac{9902}{10000}é$, with $\mu(\Omega) = 1$ in each case.

The justification of this distribution of belief-values over the é-reals is that, for the comparison between options A and B, the decision maker is happy to work in the pure real numbers; however, with respect to the comparison between options C and D, the decision maker distinguishes between the two options. For option C, she continues to think in terms of the (pure) reals, but she thinks that option D is ambiguous for similar reasons to those given in the common consequence case, and so she works in the é-reals when computing its value.

Given these belief assignments, the values of the lotteries are:

$f: (1)\bar{u}$ $\quad$ $f': (.99M + .01)\bar{u}$

$g: (.02M + .98N)\bar{u}$ $\quad$ $g': (.9902M + .0098N)\bar{u}é$

Now define: $\beta = .02M + .98N$. We suppose that $\beta < 1$, on the basis that the decision maker's utility function is concave. This implies that, in the first pairwise comparison – where the decision maker is operating entirely in the real numbers – f is chosen over g. If, however, the decision maker is sufficiently ambiguity avid, then we might obtain: $\alpha < \gamma$, where $\gamma = \left(\frac{\beta}{100} + .99M\right)/\left(\frac{1}{100} + .99M\right)$, in which case the decision maker chooses g' over f'. This accounts for the preference relations observed when the common ratio effect is in play.

The paradox is depicted in Figure 3, where the heavy dashed line describes the second pairwise choice and has a slope of $-\gamma$; and the sloped solid lines are 'indifference curves' depicting the decision maker's attitude to ambiguity, each with a slope of $-\alpha$ (the light dashed lines are for indicative purposes, and have a slope of -45°).

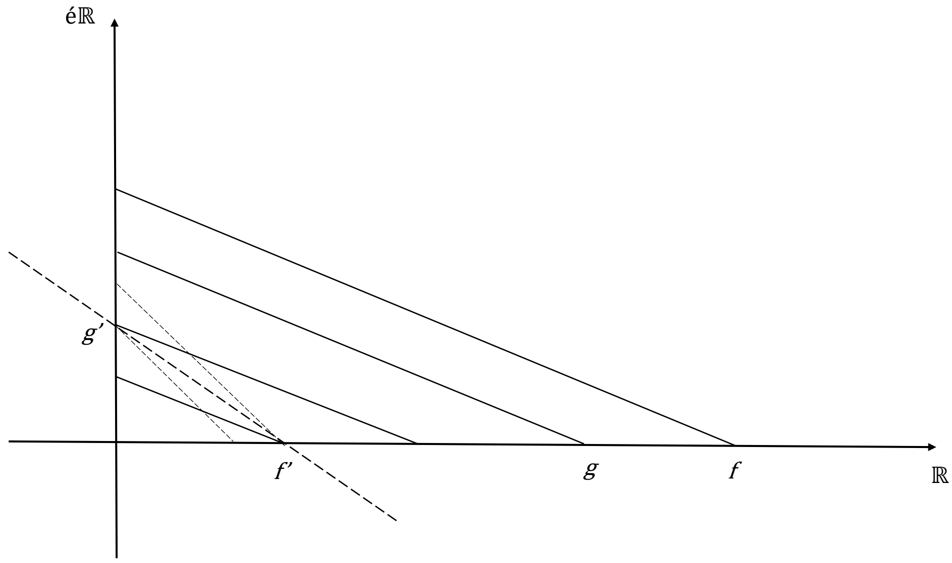

Figure 3

Part III

In this paper, we've supposed that decision makers express their beliefs on a ring of numbers which is isomorphic to $\mathbb{R} \oplus \mathbb{R}$ rather than expressing their beliefs on $\mathbb{R}$ only, as is the case in canonical decision theory. One consequence of their doing so is that ratios of numbers that have the same right angle value are treated differently by decision makers if those numbers differ in their relative real-to-é-real content and if decision makers are either ambiguity averse or ambiguity avid (i.e., if they have: $\alpha \neq 1$). Moreover, this fact is related to the non-canonical revelation of preference that's implied by the Allais paradox, as we've just seen. It was Rubinstein (1988, p.152) who first pointed out that the explanation of that paradox might lie in understanding the nature of the dissimilarities of the ratios of beliefs of decision makers. That being said, we've been able to show that a positive answer can be given to the eponymous question he posed in his paper – which is to say, there *is* a decision theory resolution to the Allais paradox *if* one is willing to extend the definition of expected utility to include expected utility defined on the specified ring.

Of course, in order to make this claim, we've had to assume that decision makers use a very definite procedure to allocate numbers into one part of the ring rather than into another part. Specifically, we've supposed that decision makers use the 'length of the fraction' when choosing to describe one number as ambiguous and another as not. It may be that other hypotheses are possible or, indeed, more plausible. More empirical work is needed here – we're still far from Rubinstein's aim of having a “descriptive theory for decisions under risk” (p.153). However, as the above arguments suggest, we do have a way for dealing with these issues within the framework of decision theory, broadly defined.

Another point where further empirical evidence would be helpful in determining how decision makers think concerns their attitude to ambiguity. In the above, we’ve assumed that their attitude is entirely captured by the parameter, α. This then implies that the indifference map over the é-real domain has the ‘parallel-linearity’ property depicted in Figures 1-3. It might be thought more plausible, however, that decision makers are more ambiguity avid at low levels of expected utility since the stakes are low, whilst at high(er) levels of expected utility they’re more ambiguity averse. This then implies that the indifference map looks like that given in Figure 4. With these preferences, the Allais-paradox behaviours are even more surely predicted than they are in the cases where indifference curves are parallel. It remains a matter for empirical investigation as whether or not decision makers’ preferences generally vary with the expected utility of outcomes in the manner described in Figure 4.

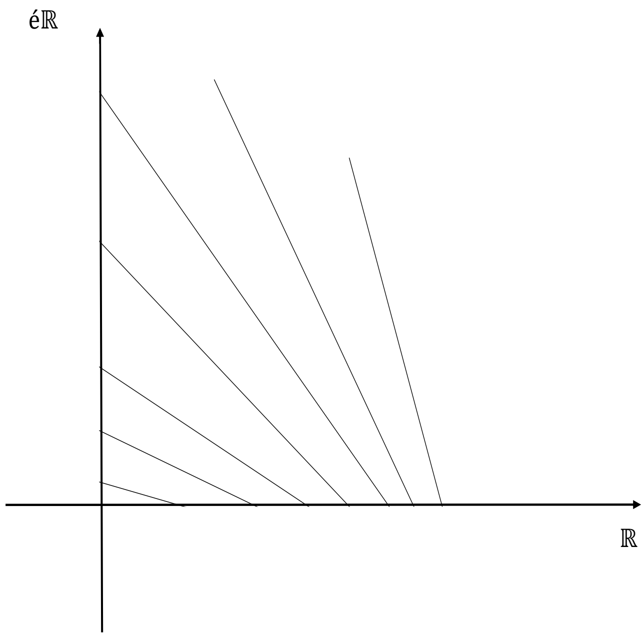

Figure 4

Appendix:

In this Appendix, we provide an axiomatic grounding for the functional model given in the text. We begin by reminding ourselves that the domain of choice for decision makers is the set of lotteries, $\mathcal{L}$, where each $L \in \mathcal{L}$ is defined as follows:

$L = (\mu_{\acute{e}}^{L}: X \to \{a + b\acute{e} \mid \lfloor a + b\acute{e} \rfloor \leq 1; 0 \leq a, b; \sum_{x \in X} \lfloor \mu_{\acute{e}}^{L}(x) \rfloor = 1\})$

The following three axioms extend the canonical von Neumann axioms to the case at hand (i.e., to the case where decision makers work with an isomorphism of $\mathbb{R} \oplus \mathbb{R}$ rather than $\mathbb{R}$).

Axiom 1: *Ordering*: there's a (complete) preference ordering, $\succsim$, over $\mathcal{L}$.

In order to define the next two axioms, we need the following definition.

Definition 3: if $\lambda = a_{\lambda} + b_{\lambda}\acute{e}$, with $0 \leq a_{\lambda}, b_{\lambda}$ and $a_{\lambda} + b_{\lambda} \leq 1$, then $\lambda^{c} = a_{\lambda^{c}} + b_{\lambda^{c}}\acute{e}$, with $0 \leq a_{\lambda^{c}}, b_{\lambda^{c}}$; $a_{\lambda^{c}} + b_{\lambda^{c}} = 1 - (a_{\lambda} + b_{\lambda})$.

Axiom 2: *Archimedean*: there are: $\theta, \theta^{c} \in \acute{E}$, with $0 \leq a_{\theta}, a_{\theta^{c}}, b_{\theta}, b_{\theta^{c}}$ and $\lfloor \theta \rfloor, \lfloor \theta^{c} \rfloor \leq 1$ such that for any lotteries, $L, L', L'' \in \mathcal{L}$ with: $L'' \succsim L \succsim L'$, we have: $\theta L'' + \theta^{c} L' \sim L$; and, for degenerate or constant lotteries: $L_{x} \equiv x$, with: $x'' \succsim x \succsim x'$, whenever we have: $\theta x'' + \theta^{c} x' \sim x$ then $\theta, \theta^{c} \in \mathbb{R} \subset \acute{E}$.

Axiom 3: *Independence*: for all lotteries, $L, L', L'' \in \mathcal{L}$, $L \neq L', L''$ and for any $\lambda \in \acute{E}$ such that: $0 \leq a_{\lambda}, b_{\lambda}$ and $a_{\lambda} + b_{\lambda} < 1$, we have: $\lambda L + \lambda^{c} L'' \succsim \lambda L + \lambda^{c} L' \Leftrightarrow L'' \succsim L'$; and, for all lotteries, $L, L', L'' \in \mathcal{L}$ and for any $\lambda \in [0,1)$, we have: $\lambda L + (1 - \lambda) L'' \succsim \lambda L + (1 - \lambda) L' \Leftrightarrow L'' \succsim L'$.

These first three assumptions are sufficient to determine a real-valued utility function over prizes or outcomes, and an expected utility function whose values lie in $\acute{E}$. This latter function provides a partial representation of the decision maker's preference ordering over lotteries. To determine a complete, real-valued representation of her preferences over lotteries, the decision maker makes explicit her attitude to ambiguity. In what follows, x^{*} is the best available prize and x_{*} the worst, with $x^{*} \succ x_{*}$; furthermore, the decision maker's attitude to ambiguity is represented by the parameter, $\alpha \in \mathbb{R}$.

Axiom 4: *Ambiguity*: whenever we have: $L \sim v.x^{*} + v^{c}.x_{*}$, where $v = a^{v} + b^{v}\acute{e}$, and $v + v^{c} = \sum_{x \in X} \mu_{\acute{e}}(x)$, then there's a unique real value: $r = a^{v} + \alpha^{-1} b^{v}$, with $0 < \alpha$, such that:

$L \sim r.x^* + (1-r)x_*$, and the mapping that accomplishes this – i.e., which sends $(v, v^c) \mapsto (r, 1-r)$ – is denoted by φ.

The following assumption follows is also fairly standard and intuitive.

Axiom 5: *Monotonicity*: whenever we have: $L' \sim r'.x^* + (1-r')x_*$ and $L'' \sim r''.x^* + (1-r'')x_*$, then $L'' \gtrsim L' \Leftrightarrow r'' \geq r'$.

Given these axioms, we can state the following.

Theorem: The five stated axioms imply that there exists a real-valued utility function, $u: X \to \mathbb{R}$, and, moreover, an ambiguity adjusted utility function exists such that, for any two lotteries: $L'' \gtrsim L' \Leftrightarrow \varphi(\sum_{x \in \mathrm{X}} \mu''_{\acute{e}}(x) \cdot u(x)) \geq \varphi(\sum_{x \in X} \mu'_{\acute{e}}(x) \cdot u(x))$.

Proof: We begin by showing that the first three axioms can be used to determine a real-valued utility function that accomplishes: $u_y \geq u_x \Leftrightarrow y \gtrsim x$, where $u_y, u_x \in \mathbb{R}$ are the values that solve: $y \sim [u_y x^* + (1-u_y)x_*]$ and $x \sim [u_x x^* + (1-u_x)x_*]$, respectively.

To see this, we note that the first two axioms (i.e., 'ordering' and 'Archimedean') imply that the two indifference relations given in the preceding sentence are well posed, with: $u_y, u_x, (1-u_y), (1-u_x) \in \mathbb{R}$. The statement that: $u_y \geq u_x \Leftrightarrow y \gtrsim x$ follows from a familiar result which states that, for $\kappa, \lambda, (1-\kappa), (1-\lambda) \in [0,1]$, we have: $\lambda x^* + (1-\lambda)x_* \gtrsim \kappa x^* + (1-\kappa)x_* \Leftrightarrow \lambda \geq \kappa$, which is itself implied by those axioms and independence (we recall that the generalized concept of independence with which we're working entails the canonical definition, so that the argument just given goes through in the usual fashion – see, for example, the proof in Mas-Colell, Whinston & Green, 1995, §6.B). This result also implies that utility is unique.

In the next part of the argument, we want to show how lotteries are evaluated, compared and chosen.

To see how, take the lottery: $L' = (\mu'_{\acute{e}}(x_1), x_1;\ \mu'_{\acute{e}}(x_2), x_2;\ \mu'_{\acute{e}}(x_3), x_3; \ldots)$, which can also be represented by the vector sum of constant lotteries: $L' = \mu'_{\acute{e}}(x_1)L_{x_1} + \mu'_{\acute{e}}(x_2)L_{x_2} + \mu'_{\acute{e}}(x_3)L_{x_3} + \cdots$.

By axiom three (independence), the earlier derivation of utilities, and by identifying constant lotteries with their outcomes: $L_{x_i} \equiv x_i$, we have:

$\mu'_{\acute{e}}(x_1)x_1 + \mu'_{\acute{e}}(x_2)x_2 + \mu'_{\acute{e}}(x_3)x_3 + \cdots \sim$

$\mu'_{é}(x_1)\left[u_{x_1}x^* + \left(1 - u_{x_1}\right)x_*\right] + \mu'_{é}(x_2)\left[u_{x_2}x^* + \left(1 - u_{x_2}\right)x_*\right] + \mu'_{é}(x_3)\left[u_{x_3}x^* + \left(1 - u_{x_3}\right)x_*\right] \ldots$

This latter expression (i.e., the expression following the indifference sign) can be more conveniently rendered by writing: $u(x_1) = u_{x_1}$, and re-arranging to yield the following statement:

$$L' \sim \left(\sum_{x \in \mathrm{X}} \mu'_{é}(x) \cdot u(x)\right) x^* + \left(\sum_{x \in \mathrm{X}} \mu'_{é}(x) \cdot \left(1 - u(x)\right)\right) x_*$$

The expressions in the parentheses define the values, $v', v^{c\prime} \in \acute{E}$, spoken of in the fourth axiom (ambiguity):

$$v' = \sum_{x \in \mathrm{X}} \mu'_{é}(x) \cdot u(x) \qquad\qquad v^{c\prime} = \sum_{x \in \mathrm{X}} \mu'_{é}(x) \cdot \left(1 - u(x)\right)$$

On the supposition stated there, there's a mapping, φ, that sends: $v' \mapsto r'$, $v^{c\prime} \mapsto 1 - r'$, so as to attain: $L' \sim v'x^* + v^{c\prime}x_* \sim r'x^* + (1 - r')x_*$.

More expansively, we have:

$$r' = \varphi\left(\sum_{x \in \mathrm{X}} \mu'_{é}(x) \cdot u(x)\right)$$

and:

$$L' \sim \varphi\left(\sum_{x \in \mathrm{X}} \mu'_{é}(x) \cdot u(x)\right) x^* + \left(1 - \varphi\left(\sum_{x \in \mathrm{X}} \mu'_{é}(x) \cdot u(x)\right)\right) x_*$$

To complete this phase of the argument, we want to show that comparative decisions are made in the manner proposed. To see that this is so, suppose that there's another lottery, L'':

$$L'' \sim \varphi\left(\sum_{x \in \mathrm{X}} \mu''_{é}(x) \cdot u(x)\right) x^* + \left(1 - \varphi\left(\sum_{x \in \mathrm{X}} \mu''_{é}(x) \cdot u(x)\right)\right) x_*$$

Then, by the last axiom (monotonicity), we have:

$$\varphi\left(\sum_{x \in \mathrm{X}} \mu''_{é}(x) \cdot u(x)\right) \geq \varphi\left(\sum_{x \in \mathrm{X}} \mu'_{é}(x) \cdot u(x)\right) \Leftrightarrow L'' \gtrsim L'$$

As: $\varphi(\sum_{x\in X}\mu''_{é}(x)\cdot u(x)) \geq \varphi(\sum_{x\in X}\mu'_{é}(x)\cdot u(x)) \Leftrightarrow \phi(\sum_{x\in X}\mu''_{é}(x)\cdot u(x)) \geq \phi(\sum_{x\in X}\mu'_{é}(x)\cdot u(x))$, the functional form given in Part I is warranted by the stated axioms, which is what we intended to show.

References

Allais, M. (1953) "Le comportement de l'homme rationnel devant le risque: Critique des postulats et axiomes de l'ecole américaine" *Econometrica* 21(4): 503-46.

Izhakian, Y. & Izhakian, Z. (2014) "Decision making in phantom spaces" *Economic Theory*, DOI 10.1007/s00199-013-0798-3.

Mas-Colell, A., Whinston, M., & Green, J. (1995) *Microeconomic Theory*, OUP: Oxford.

Rubinstein, A. (1988) "Similarity and decision-making under risk (is there a decision theory resolution to the Allais paradox?)" *Journal of Economic Theory*, 46: 146-53.

Sharpe, K. (2015) "On the Ellsberg paradox and its extension by Machina", *mimeo*, http://dx.doi.org/10.2139/ssrn.2630471 (an earlier version of this paper was read to the Australian Conference of Economists, 8 July, 2015).