A Small State-Space Model of the Australian Economy

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ABSTRACT

Using a small state space macroeconomic model, we apply maximum likelihood estimation and the Kalman filter to obtain joint estimates of the unobservable medium-run paths of potential output and its normal rate of growth, the natural rate of unemployment and the natural real interest rate. Our unobserved component analysis indicates that in 2005, the Australian economy was close to potential output, the normal rate of growth was low, unemployment was below the natural rate presaging inflationary pressures, and that the real rate of interest was significantly below its natural rate, suggesting that monetary policy was too expansionary.

JEL Classification: E32, C32
KEYWORDS: Natural rates, Kalman filter, state space model, unobserved components

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1. Introduction

A traditional view of business cycles is that they are short-run stochastic movements of real variables around their smoothed trend values. These smoothed trends or natural rates play an important role in macroeconomic models as benchmarks to compare with current values. Wicksell (1936) introduced the concept of the natural interest rate, and more recently, there has been a revival in the literature on the subject following Woodford's (2003) book. In contemporary terms, the natural rate of interest is the equilibrium real rate (sometimes called the neutral rate) that would arise if wages and prices were completely flexible, given current factors. Phelps (1967) and Friedman (1968) introduced the related idea of the natural rate of unemployment. More specifically, the tradeoff between inflation and unemployment is temporary, so that the actual unemployment rate converges to the natural rate, at which point the inflation rate remains constant. Thus this benchmark unemployment rate is also known as the non-accelerating inflation rate of unemployment (NAIRU). When the economy is anchored at the NAIRU, GDP must be at the natural level of output, which is sometimes called the level of potential output. A short-run output gap emerges if GDP deviates from the natural level of output. However this potential output need not be constant in the medium run—given productivity growth and factor accumulation, there will be a normal rate of growth of potential output in the medium run. As the horizon progresses to the long run, this growth rate becomes the steady-state growth rate of the economy. Short-run deviations from the natural (and normal) rates in the medium run can be explained by the presence of imperfect information (e.g. Lucas (1972)) or nominal rigidities. These deviations affect movements in aggregate demand and supply, which in turn stimulate adjustment processes to return the economy to the medium-run equilibrium.

These natural rate and level concepts are central to the conduct of monetary policy. An inflation-targeting central bank needs to assess the level of economic activity variables in relation to their natural values to judge the pressures on inflation relative to its target and on any other target variables. When output grows faster than normal and exceeds its potential value, the unemployment rate will fall short of the NAIRU, wage inflation will rise, the real interest rate will be below its natural level, and so there will be upward pressure on inflation. The central bank is likely to tighten monetary policy to steer inflation and output back to their target and natural values. The short-term interest rate rises from its current position until the medium-run equilibrium is restored, and monetary policy returns to its neutral stance.

Although these natural economic indicators provide useful information to economists and policymakers, they are unobservable by nature and must be inferred from the data. The objective of this paper is to estimate a multivariate, unobserved components (UC) model for the Australian economy that allows for the simultaneous estimation of the paths of potential output and its normal growth rate, the NAIRU and the natural real interest rate. The multivariate state space model comprises a dynamic IS equation of the output gap representing aggregate demand (AD), an expectations-augmented Phillips curve that represents aggregate supply (AS), an Okun's relation connecting cyclical movements of output to unemployment, and a first order condition from intertemporal optimization giving the medium run relationship between the real interest rate and output growth . The model is estimated using maximum likelihood over the period 1984Q1 to 2005Q4, extracting unobservable state variables with the Kalman filter. Inflation information from the AS relation is used to infer the output gap (defined as the difference between actual and potential output), which in turn through the dynamic IS equation is used to infer the real interest rate gap (defined as the difference between actual short-term interest rate and its natural rate) and the natural real rate of interest. An Okun's equation relates the cyclical fluctuations in the product market (the output gap) to the cyclical fluctuations in the labour market (hence inferring the unemployment gap).

The paper proceeds as follows. In section 2, different univariate and multivariate measures of natural rates are discussed. This discussion motivates the multivariate model outlined in section 3. Section 4 describes the data, some econometric issues related to Kalman filter and presents the parameter estimates and the multivariate UC smoothed natural rates. Section 5 finishes the paper with some concluding remarks.

2. Univariate and Multivariate Measures of Natural Rates

A widely used procedure to decompose macroeconomic variables (such as real output) into a trend (or potential output) and a cyclical (or the output gap) components is the Hodrick and Prescott (1997) filter.[1] The smoothness of the Hodrick-Prescott (or HP) stochastic trend depends on the input value of an *ad-hoc* smoothness parameter. If the value of the exogenous parameter is set to zero, the trend component and the actual series match each other; if the value of the parameter goes to infinity, the trend component approaches a linear deterministic trend. Baxter and King (1999) derived an estimate of the output cycle by passing the data through a filter that pre-specifies the relevant frequencies for the cycle and thus its persistence. Their approximate band-pass filter defines the cycle as having spectral power in the range between 6 and 32 quarters.

Pure statistical methods that simply 'let the data speak' do not include potentially useful information about the supply side of the economy and the business cycle contained in macroeconomic relationships such as the Phillips curve, Okun's law, and other

[1] The original paper appeared in 1980 as a Carnegie-Mellon discussion paper, and was eventually published unchanged in 1997.

indicators such as output capacity utilization.[2] Laxton and Tetlow (1992) proposed a multivariate extension to the univariate HP filter by conditioning the computation of the level of potential output on additional economic relationships. Boone *et al*. (2000) applied a multivariate HP filter to estimate the level of potential output for twenty one OECD countries. To estimate Australian potential output, de Brouwer (1998) incorporated information from inflation, unemployment and capacity utilization. Gruen *et al.* (2005) conditioned their estimates on the Phillips curve using real-time output data.

Another class of models—known as the unobserved components (UC) model—offers two advantages over the multivariate HP filter: (1) it permits a more complex system of dynamics; and (2) estimation is relatively more straightforward with the structural parameters estimated by maximum likelihood and the unobserved variables (or natural rates) derived by the Kalman filter.[3]

Within the UC framework, several papers attempted to estimate natural rates using different macroeconomic relationships. Clark (1989) and Kuttner (1991) used an Okun's equation, which defines the level of observable unemployment as a function of the unobservable output gap, to derive the level of U.S. potential output.[4] In Kuttner (1994), the level of potential output is extracted from information contained in the Phillips curve so that the economy maintains a constant level of inflation. The constant-inflation natural unemployment rate (or the NAIRU) is provided by King *et al.* (1995), Staiger *et al.* (1996) and Laubach (2001) using a similar framework.

Instead of conducting partial analysis on potential output or the NAIRU, it is possible to estimate a system of equations that features the Phillips curve, which imposes

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[2] While there are many other univariate filters which may have better properties, our focus is on the gains from multivariate extensions.

[3] The multivariate HP filter implemented by Laxton and Tetlow (1992) is a two-step procedure. First, the economic relationships are separately estimated. The regression residuals are inserted into the multivariate HP minimization problem to estimate the unobservable variable. This two-step procedure is repeated with several iterations until convergence is achieved. See Boone (2000) for more details.

[4] In Kuttner (1991), the Okun's equation relates the change in unemployment to the output gap.

a constant-inflation restriction on the path of potential output or the NAIRU, and incorporating the covariation restrictions on cyclical output and cyclical unemployment through the Okun's relation. Some examples of papers that model the mutual dependency of output and unemployment include Apel and Jansson (1999a, 1999b) and Benes and N'Diaye (2004).

In addition to an Okun's relation that represents the linkages between output and unemployment, movements from the real interest rate relative to its natural rate can be embedded in the IS relation for the ouput gap describing product market equilibrium. The incorporation of this extra channel is likely to enhance the estimation of the cyclical paths of unobservable variables in the economy.

The natural real interest rate is likely to vary over time, and in an intertemporally optimal setting will be determined by factors such as underlying productivity growth and the rate of time preference. For example, Laubach and Williams (2003) found substantial variations in the natural interest rate over the past four decades in the U.S. They also suggested that there is an approximate one-for-one relationship between natural rate variation and changes in the growth rate of potential GDP.

3. A Multivariate Model of Unobserved Components

Both output and unemployment are decomposed in (1) and (2) into a stochastic trend component and the stochastic cyclical variations around this trend. The trend components are taken to be the level of potential output and the natural rate of unemployment (or the NAIRU) that are associated with the medium-run equilibrium when prices and wages have fully adjusted to shocks. When demand or supply shocks occur, there will be deviations from the trend component values in the short run because of nominal rigidities, and these are defined as the output and unemployment gap measures.

$$y_t = y_t^* + \tilde{y}_t \tag{1}$$

$$u_t = u_t^* + \tilde{u}_t \tag{2}$$

In (1) and (2), y_t is the log of real GDP, y_t^* is the log of potential GDP and $\tilde{y}_t$ denotes the output gap; u_t is the unemployment rate, u_t^* is the NAIRU value, and $\tilde{u}_t$ represents the unemployment gap. Note that all variables are potentially time-dependent.

Following Rudebusch and Svensson (1999), the aggregate demand side of the economy is described by a reduced-form IS equation as in (3):

$$\tilde{y}_t = a_1 \tilde{y}_{t-1} + a_2 \tilde{y}_{t-2} + a_3 \left(r_{t-8} - r_{t-8}^* \right) + a_4 \Delta LTOT_{t-4} + a_5 \tilde{y}_t^{US} + \varepsilon_t^{\tilde{y}} \tag{3}$$

A stationary AR(2) process is specified for the dynamics of the output gap ($\tilde{y}_t$), which follows Watson (1986) and Clark (1987) for the U.S. economy and de Brouwer (1998) for Australia. As in Laubach and Williams (2003), a real interest rate gap $(r_{t-j} - r_{t-j}^*)$ is included in the output gap equation. After preliminary OLS estimations using general to specific tests, we found that the 8th lag (2 years) of the real interest rate gap should be included. Stone *et al*. (2005) find it necessary to include lags 1 to 7 of the same variable for the effect of a change in monetary policy on the output gap. To capture the positive effect that a rising terms of trade has on the output gap, the fourth lag of the quarterly change of the terms of trade ($\Delta LTOT_{t-4}$) is included. Given Australia is a small open economy, the contemporaneous value of the U.S. output gap ($\tilde{y}_t^{US}$) obtained from a Hodrick-Prescott filter is included to explain the influence of foreign economic activity on domestic output.

Inflationary dynamics (π_t) are specified by (4) as an expectations-augmented Phillips curve:

$$\pi_t = b_1\pi_{t-1} + b_2\pi_{t-2} + b_3\tilde{y}_{t-1} + b_4\pi_{t-1}^{imp} + b_5\pi_t^e + \varepsilon_t^{\pi} \qquad (4)$$

where the process of expectations is assumed to be driven by lagged inflation (π_{t-1} and π_{t-2}), lagged imported price inflation (π_{t-1}^{imp}) and contemporaneous consumer survey expected inflation (π_t^e) as advocated by Roberts (1997). The influence of excess demand comes from the lagged output gap, which reflects the nominal inertia of price responses to economic conditions.

The connection between the unemployment gap ($\tilde{u}_t$) and the output gap is represented by an Okun equation (5).

$$\tilde{u}_t = c_1\tilde{u}_{t-1} + c_2\tilde{u}_{t-2} + c_3\tilde{y}_t + \varepsilon_t^{\tilde{u}} \qquad (5)$$

Some degree of persistence in the dynamics is captured by the presence of the lagged values of the unemployment gap.

Equations (6) through (10) describe the properties of the unobservable trends in the model. Potential output is modelled by (6) as a local linear trend, where the drift term μ_{t-1} represents the trend growth rate and is a random walk process (7):[5]

$$y_t^* = y_{t-1}^* + \mu_{t-1} + \varepsilon_t^{y*} \qquad (6)$$

$$\mu_t = \mu_{t-1} + \varepsilon_t^{\mu} \qquad (7)$$

y_t^* is potential output and μ_t denotes the trend growth rate.

From household intertemporal utility maximization evaluated in the medium run,[6] we posit a relationship that links the evolution of the natural interest rate to movements in

[5] Since the drift term is assumed to be I(1), this implies that potential output is I(2).

[6] For an infinite horizon representative consumer model, where period utility is $\frac{C_t^{1-1/\sigma}}{1-1/\sigma}$ with σ being the intertemporal elasticity of substitution, and β is the rate of time preference, the first order condition is $\frac{\mu_{c,t}}{\sigma} = \ln\beta + \ln(1+r_t)$ where $\mu_{c,t}$ is the growth rate of consumption. We do not insert the implied restrictions

the trend growth rate of output (see Laubach and Williams (2003)). This is shown in (8) where r_t^* is the natural rate of interest, d captures the inverse of the elasticity of intertemporal substitution, and z_t is a random walk process as in (9) that represents other possible determinants such as the rate of time preference and risk premia.

$$r_t^* = d\mu_t + z_t \tag{8}$$

$$z_t = z_{t-1} + \varepsilon_t^z \tag{9}$$

Lastly, the natural rate of unemployment follows a pure random walk:

$$u_t^* = u_{t-1}^* + \varepsilon_t^{u*} \tag{10}$$

We assume that all innovations $\varepsilon = (\varepsilon_t^{\tilde{y}}, \varepsilon_t^{\pi}, \varepsilon_t^{\tilde{u}}, \varepsilon_t^{y*}, \varepsilon_t^{\mu}, \varepsilon_t^{z}, \varepsilon_t^{u*})'$ are i.i.d. normally distributed with zero mean and finite variances. They are additionally serially uncorrelated and contemporaneously uncorrelated innovations.

4. Data and Empirical Results

4.1 Data

The quarterly data span starts from 1984:1 to 2005:4. All data unless otherwise specified were obtained from the ABS. y_t and u_t are the Australian real GDP and unemployment rate. Domestic inflation is calculated as the year-ended change in the log of the Treasury underlying CPI which is spliced in September 1999 to the headline CPI. The same procedure is applied to compute the import inflation rate which is based on the log of the

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from this intertemporal optimizing condition into the short run product market equilibrium condition (3) on the grounds that many households are unable to optimize in the short run. However in the medium run, most will find a way to approach their optimal consumption trajectory. In the medium run equilibrium of the open economy, the current account to GDP ratio will be constant, as will be the consumption to GDP ratio, i.e. $\mu_{c,t} = \mu_t$. Therefore, at low interest rates, the medium run relation between the real interest rate and output growth is approximated by $r_t^* = \frac{\mu_t}{\sigma} - \ln\beta$.

import chain price index. The real interest rate is the nominal cash rate less the inflation rate, i.e. $r_t = i_t - \pi_t$. Estimates of U.S. potential output are available from the Congressional Budget Office (CBO) that allows us to derive the U.S. output gaps without having to construct them. Quarterly changes in the log of the terms of trade index are used since they offer higher explanatory power in preliminary OLS estimation. Inflation expectations are consumers' inflation expectations measured by the Melbourne Institute as the median expected inflation rate for the year ahead.

4.2 Estimation Issues

Before proceeding with estimation, the multivariate UC model is cast in the state-space form (see the appendix). Parameters are estimated by maximum likelihood and the natural rates are extracted by using the Kalman filter. The Kalman filter is a recursive algorithm that sequentially updates a linear projection of a dynamic system. In each period the Kalman filter provides the (one-sided) optimal predictions of the natural rates for that period conditional on information available up to and including the current period. Once the filtered natural rates are obtained, it is possible to 'smooth over' the natural estimates that are conditioned on information from the full sample; therefore the smoothed natural rates can be thought of as two-sided estimates. Two estimation issues related to the Kalman filter are discussed below, namely the choice of the initial values of the state vector and covariance matrix and the estimation of the innovation variances.

To set the initial values for the state vector, the gap measures are assumed to be stationary and so a value of zero is assigned to them. For the natural rate measures, the initial value is set to the value of the first observation of the associated variable. The dynamics of the multivariate UC model are non-stationary because the trend equations are specified to be random walks. Therefore we follow the usual practice of assigning diffuse priors to the diagonal elements of the initial state covariance matrix.[7]

The so-called "pile-up" problem identified by Stock (1994) prevents efficient estimation some parameters by direct maximization of the log-likelihood function. This is because estimates of low true variances of the innovations entering the unobservable variables dynamics are biased towards zero because a large amount of probability in their distribution piles-up at zero.[8] One way to tackle this issue is the median-unbiased estimator method developed by Stock and Watson (1998) which consists of estimating the signal-to-noise ratios in the first step and imposing these in the second estimation step. Laubach and Williams (2003) followed this approach. Instead, we shall follow King *et al*. (1995), Staiger *et al*. (1996), and Laubach (2001) in fixing the signal-to-noise ratio then testing them statistically in reference to a baseline model.

4.3 Results

Table 1 displays the parameter estimates of the multivariate UC model. Kalman smoothed estimates of potential output and its normal growth rate, the NAIRU and the natural real interest rate as well as the related gap measures are shown in Figures 1 to 7.

In the baseline model, the standard deviation of potential output (σ_{y*}) is statistically not different from zero, which indicates that the normal rate of growth of potential output is a random walk. The time-varying shock to the natural interest rate (σ_z) is statistically not different from zero, which suggests a constant rate of time preference and risk premia.

[7] We find that it is necessary to begin the recursion with a diffuse prior of zero to tie the estimated trends to a path that would run through the data. The result is that the filtered and smoothed estimates are very close to the first observation of the variable.
[8] The pile-up problem stems from discontinuity in the distribution of the state variables. Assuming a state variable following a random walk without drift, if the variance of its innovation is equal to zero, the state variable is stationary (in fact a constant) and this explains the discontinuity in the distribution function.

Given possible pile-up problem, we explored a range of fixed values for σ_z.[9] We only report σ_z = 0.005 as it produces the highest log-likelihood.[10] All of the estimated coefficients, in both cases, have the expected sign but the statistical significance is higher when σ_z = 0.005, and in particular for parameters a_2 and a_3. The sum of the autoregressive parameter estimates of the IS equation (3), the Phillips curve equation (4), and the Okun equation (5) are each less than one, which is necessary for the stationary dynamics of the output gap, domestic inflation, and the unemployment gap.

In Figures 1 and 2, the pattern of potential output[11] indicates a brief period of expansion at the end of the 1980s. This is followed by a period of excess supply covering much of the 1990s. In comparison to the HP potential output measure, the multivariate UC measures suggest that the Australian economy headed into contraction in 1990Q3, earlier by two quarters. In addition, the multivariate UC contraction is deeper and more persistent, attaining its trough in 1991Q4 at -5.0% as opposed to the HP -2.4%. Towards the end of the sample period, however, the multivariate UC output gap disagrees with the HP measure in that the economy was in a period of excess demand. This shows the merits of conditioning the path of potential output by incorporating information from the aggregate supply side through the Phillips curve and Okun's law, and from the aggregate demand side through the dynamic IS curve with intertemporal optimization .

[9] These are σ_z = 0.002, 0.003, 0.004, and 0.005.
[10] As a check of consistency with the data, the likelihood-ratio (LR) test produces a p-value = 0.052 that does not indicate a rejection of the null that the data is consistent with the restriction. When moving to a σ_z = 0.006, the p-val for the LR test is 0.018.
[11] In all the figures, the UC estimates for the baseline case is shown as a dashed line. This is provided to see the role of the restriction, but it will not be discussed further.

Table 1: Estimation Results of the Multivariate UC Model ((1) through (10))

	Baseline	**σ_z =0.005**
a_1	1.044	0.676
	(0.000)	(0.000)
a_2	-0.195	-0.218
	(0.060)	(0.017)
a_3	-0.058	-0.187
	(0.168)	(0.002)
a_4	0.088	0.087
	(0.020)	(0.017)
a_5	0.074	0.223
	(0.231)	(0.013)
b_1	1.134	1.136
	(0.000)	(0.000)
b_2	-0.279	-0.271
	(0.003)	(0.003)
b_3	0.040	0.047
	(0.027)	(0.039)
b_4	0.022	0.021
	(0.000)	(0.000)
b_5	0.096	0.088
	(0.001)	(0.001)
c_1	1.373	1.340
	(0.000)	(0.000)
c_2	-0.559	-0.517
	(0.000)	(0.000)
c_3	-0.109	-0.117
	(0.000)	(0.000)
d	3.302	5.386
	(0.402)	(0.005)
$\sigma_{\tilde{y}}$	0.006	0.005
	(0.000)	(0.000)
σ_{π}	0.003	0.003
	(0.000)	(0.000)
$\sigma_{\tilde{u}}$	-0.001	-0.001
	(0.002)	(0.012)
σ_{y*}	0.000	0.000
	(0.999)	(0.999)
σ_{μ}	0.001	0.001
	(0.001)	(0.001)
σ_{u*}	0.002	0.002
	(0.000)	(0.000)
σ_z	0.000	
	(0.999)	
Log likelihood	1060.79	1062.68

Note: p-values are given in parentheses.

Figure 1: Real GDP and Smoothed Potential Output

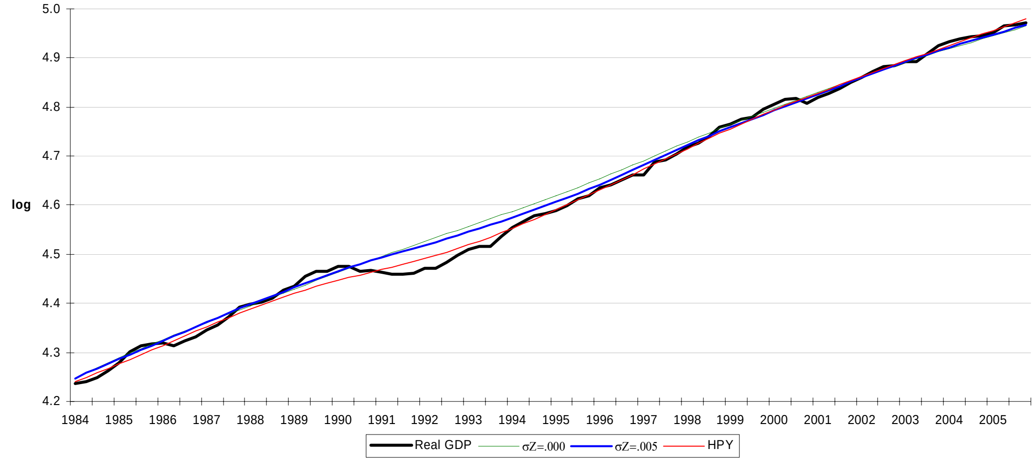
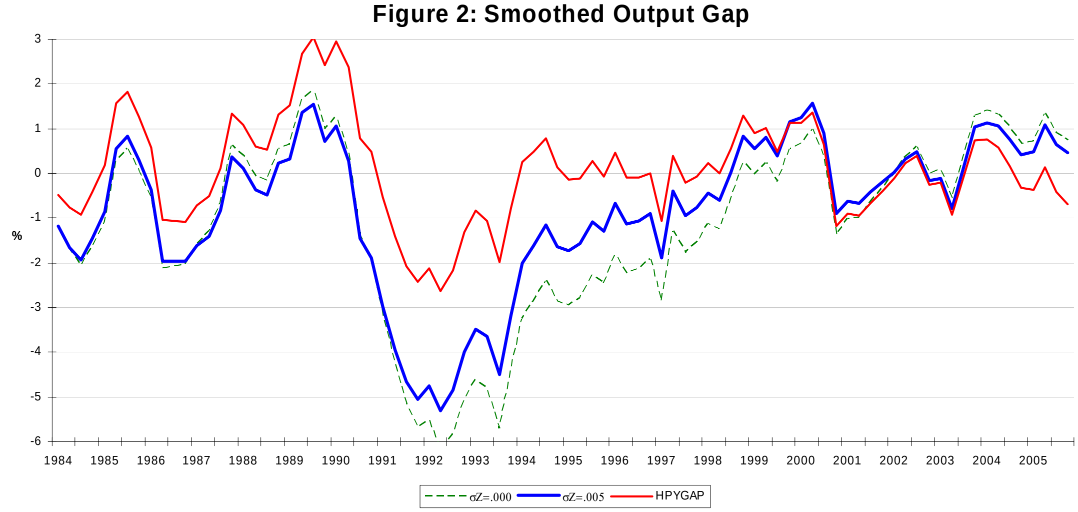


Figure 2: Smoothed Output Gap



We show in figure 3 that the NAIRU fell throughout the sample period in general, except for the temporary pickups around the 1990-91 recession. Unlike the HP filter which essentially plots the trend line through the unemployment data, the multivariate

NAIRU was much lower than its HP counterpart for most of the time. Even though the unemployment rate has been on a downward trend since the early 1990s, the result suggests that the accompanying decline in the NAIRU has buffered the Australian economy from inflationary pressure. Mirroring the estimates of the output gap, the unemployment gap values in Figure 4 show that the slack conditions in the labour market persisted for much of the 1990s, despite the HP measure suggesting tightness after 1994. Towards the end of 2005, unlike the HP filter, the multivariate UC measure indicates that the labour market had become increasingly tight.

Figure 3: Unemployment and Smoothed NAIRU

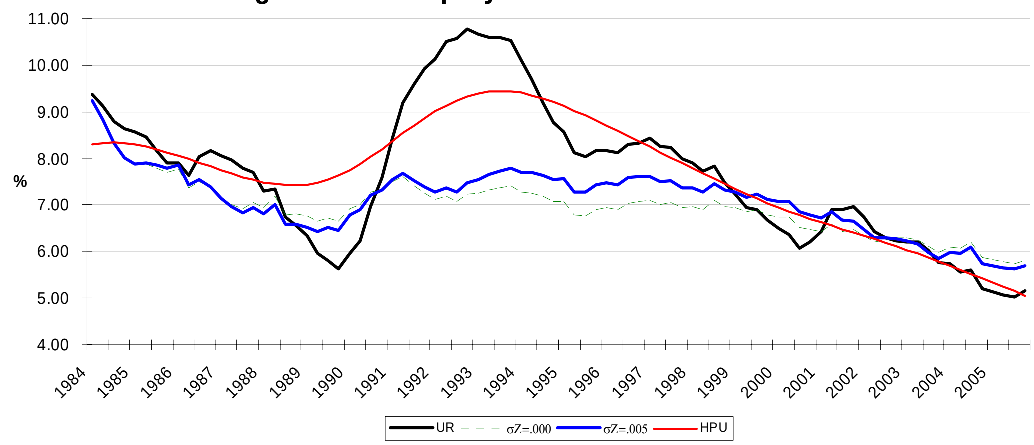
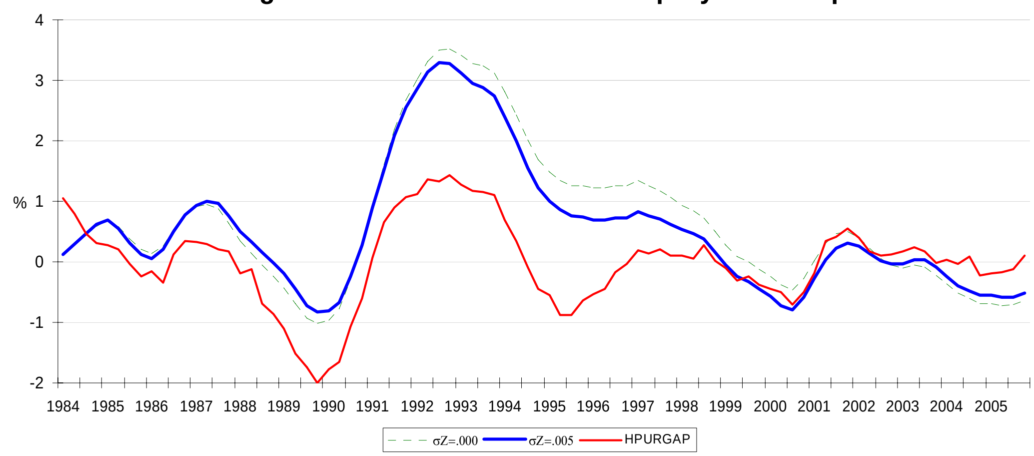


Figure 4: The Smoothed Unemployment Gap



The HP and multivariate UC filter provide contrasting perspectives on the monetary policy stance over the sample period. As seen in Figures 5 and 6, the univariate HP estimates suggest that the natural real interest rate was significantly higher than the UC case until 1999, but the roles reverse after 1999. The HP real interest rate gap measures indicate that monetary policy became expansionary from 1987 after the stock market crash and for about three years after the recession in 1991, but was quite restrictive between the two expansionary phases. On the other hand, the multivariate UC measure suggests that monetary policy was highly contractionary for most of the sample period before being reversed after the Asian financial crisis. During the severe monetary policy contraction in 1989-90, the actual real cash rate went up dramatically, and the HP measure followed it to a degree in its economic blindness. In 1989-90, the UC real rate fell, which is what economic insight would suggest. Since output and consumption growth fell in that recession, the medium run real interest rate had to follow suit to a degree to maintain intertemporal balance. Since 1999, the UC real rate suggests monetary policy has stayed relatively stimulative with the natural real interest rate converging on 4% at the end of 2005.

Figure 5: Real Cash Rate and the Smoothed Natural Rate

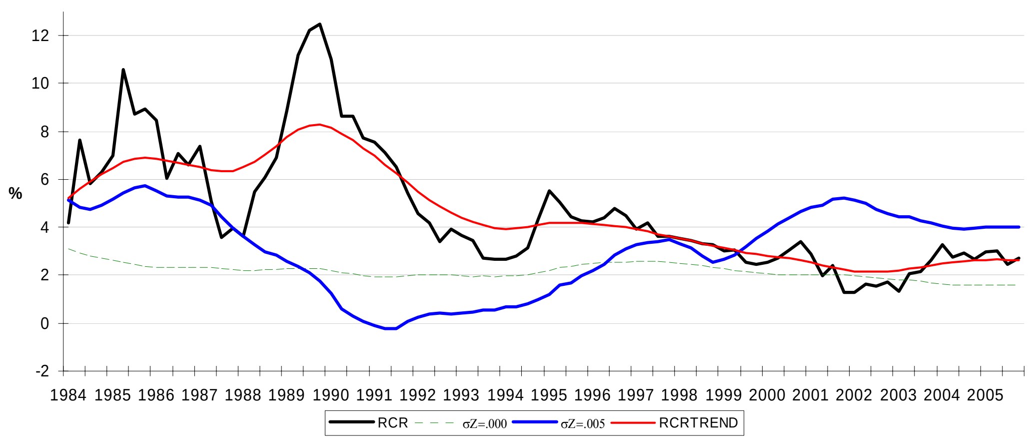
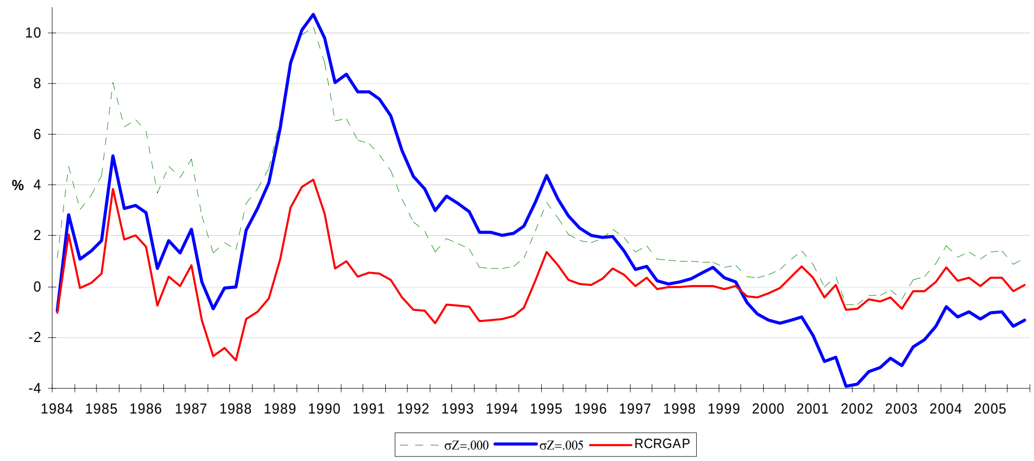
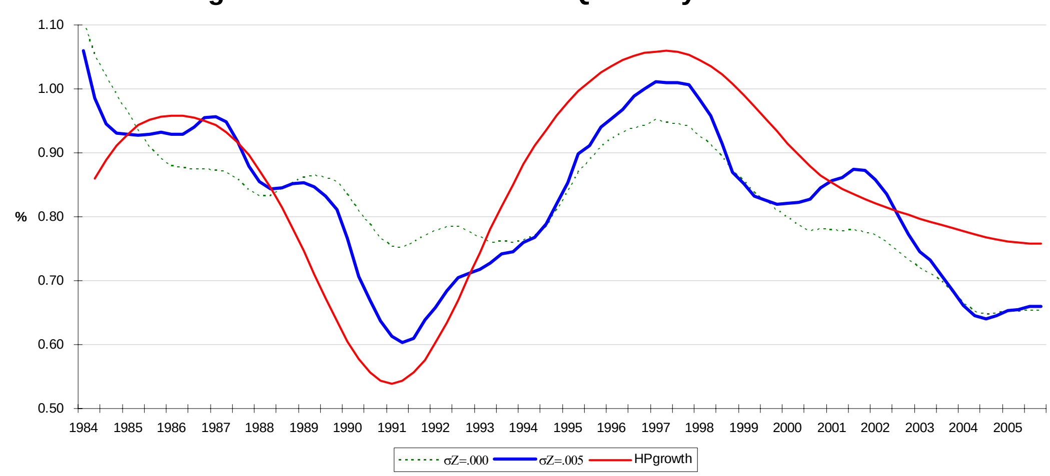


Figure 6: Smoothed Real Interest Rate Gap



Finally, the estimates of the underlying normal quarterly growth rate of the economy are shown in Figure 7. The HP estimates exaggerates the movements and appears smoother than the UC ones. The fall in normal growth rates began well before the onset of the 1990-1 recession, but turned at the bottom of the recession. The normal rate of growth has actually declined significantly since 1996, despite output growth being consistently stable and positive. In 2005 the normal growth rate was almost as low as it had been in the 1991 trough.

Figure 7: Smoothed Normal Quarterly Growth Rate



Concluding Comments

Natural rates and normal growth rates are medium-run benchmarks that permit a judgement about whether the actual rates are too high, too low or just right. We have jointly estimated the time paths of these unobservable benchmarks using maximum likelihood methods and the Kalman filter for Australian data from 1984 to 2005. We constructed a standard macroeconomic model for inflation, output and unemployment, and our parameter estimates were all significant with the expected signs. From the inferred natural rate of unemployment, the natural level of output (or potential output) and its normal growth rates, and the natural real rate of interest, we have been able to assess the state of the actual economy and comment on the stance of monetary policy over the two decades of the sample. At the end of 2005, we conclude that output was close to potential, that its normal growth rate was actually quite low at about 2.6% on an annualized basis, that unemployment was about a ½ percentage point below its natural rate of 5.7%, and that the real cash rate was actually 1.3 percentage points below its natural rate of 4%. This suggests that monetary policy was still expansionary, but it is not clear that this could have any beneficial effect with output at potential, and with monetary policy expected to be neutral in regard to the normal rate of growth.

Appendix

The state space representation is consisted of a measurement equation:

$$\mathbf{w}_t = \mathbf{F}\boldsymbol{\xi}_t + \mathbf{v}_t \tag{11}$$

and a state equation:

$$\boldsymbol{\xi}_t = \mathbf{G}\boldsymbol{\xi}_{t-1} + \mathbf{H}\mathbf{x}_t + \boldsymbol{\upsilon}_t \tag{12}$$

where $\mathbf{w}_t$ is the vector of observable variables, $\boldsymbol{\xi}_t$ is the vector of state (or unobservable variables), and $\mathbf{x}_t$ is the vector of exogenous variables. $\mathbf{v}_t$ and $\boldsymbol{\upsilon}_t$ are white noise innovation vectors.

The measurement equations in matrix form:

$$\begin{bmatrix} y_t \\ u_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{y}_t \\ \tilde{y}_t \\ \pi_t \\ \pi_{t-1} \\ \tilde{u}_t \\ \tilde{u}_{t-1} \\ y_t^* \\ \mu_t \\ r_t^* \\ r_{t-1}^* \\ r_{t-2}^* \\ r_{t-3}^* \\ r_{t-4}^* \\ r_{t-5}^* \\ r_{t-6}^* \\ r_{t-7}^* \\ z_t \\ u_t^* \end{bmatrix} \tag{13}$$

The transition equations in matrix form:

$$
\left[\begin{array}{c}
\tilde{y}_t \\
\tilde{y}_{t-1} \\
\pi_t \\
\pi_{t-1} \\
\tilde{u}_t \\
\tilde{u}_{t-1} \\
y_t^* \\
\mu_t \\
r_t^* \\
r_{t-1}^* \\
r_{t-2}^* \\
r_{t-3}^* \\
r_{t-4}^* \\
r_{t-5}^* \\
r_{t-6}^* \\
r_{t-7}^* \\
z_t \\
u_t^*
\end{array}\right]
=
\left[\begin{array}{cccccccccccccccccc}
a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_3 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b_3 & 0 & b_1 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_3 a_1 & c_3 a_2 & 0 & 0 & c_1 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_3 a_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\left[\begin{array}{c}
\tilde{y}_{t-1} \\
\tilde{y}_{t-2} \\
\pi_{t-1} \\
\pi_{t-2} \\
\tilde{u}_{t-1} \\
\tilde{u}_{t-2} \\
y_{t-1}^* \\
\mu_{t-1} \\
r_{t-1}^* \\
r_{t-2}^* \\
r_{t-3}^* \\
r_{t-4}^* \\
r_{t-5}^* \\
r_{t-6}^* \\
r_{t-7}^* \\
r_{t-8}^* \\
z_{t-1} \\
u_{t-1}^*
\end{array}\right]
+
\left[\begin{array}{c}
a_3 r_{t-8} + a_4 \Delta LTOT_{t-4} + a_5 \tilde{y}_t^{US} \\
0 \\
b_4 \pi_{t-1}^{imp} + b_5 \pi_{t-1}^{e} + b_6 \pi_{t-2}^{e} \\
0 \\
c_3 \left(a_3 r_{t-8} + a_4 \Delta LTOT_{t-4} + a_5 \tilde{y}_t^{US} \right) \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}\right]
+
\left[\begin{array}{c}
\varepsilon_t^{\tilde{y}} \\
0 \\
\varepsilon_t^{\pi} \\
0 \\
c_3 \varepsilon_t^{\tilde{y}} + \varepsilon_t^{\tilde{u}} \\
0 \\
\varepsilon_t^{y^*} \\
\varepsilon_t^{\mu} \\
d\,\varepsilon_t^{\mu} + \varepsilon_t^{z} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\varepsilon_t^{z} \\
\varepsilon_t^{u^*}
\end{array}\right]
\quad (14)
$$

The covariance matrix of the residuals of the transition equations is as follows:

$$Q=\begin{bmatrix}
\sigma_{\tilde{y}}^2 &&&&&&&&&&&&&&&&& \\
0 & 0 &&&&&&&&&&&&&&&& \\
0 & 0 & \sigma_{\pi}^2 &&&&&&&&&&&&&&& \\
0 & 0 & 0 & 0 &&&&&&&&&&&&&& \\
c_3\sigma_{\tilde{y}}^2 & 0 & 0 & 0 & c_3^2\sigma_{\tilde{y}}^2+\sigma_{\tilde{u}}^2 &&&&&&&&&&&&& \\
0 & 0 & 0 & 0 & 0 & 0 &&&&&&&&&&&& \\
0 & 0 & 0 & 0 & 0 & 0 & \sigma_{y*}^2 &&&&&&&&&&& \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{\mu}^2 &&&&&&&&&& \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d\sigma_{\mu}^2 & d\sigma_{\mu}^2+\sigma_{z}^2 &&&&&&&&& \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &&&&&&&& \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &&&&&&& \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &&&&&& \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &&&&& \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &&&& \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &&& \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 && \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{z}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{z}^2 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{u*}^2
\end{bmatrix} \qquad (15)$$

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