Treasure game*

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Abstract

We study the investment choice of two agents in a R&D race where the necessary threshold investment for success is uncertain. The race is modeled as a multistage game with observed previous actions where the player's probability of success depends only on his investment in that period. We are able to characterize a symmetric Markov equilibrium for n players and find that any search with $n \geq 2$ players is *always* less efficient than one-player search. There are two sources of this inefficiency: more players search too fast, more players do not take all efficient projects.

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1 Introduction

The R&D literature has grown up substantially in recent years. There are three important directions.

(i) The classical papers, Loury (1979), Dasgupta and Stiglitz (1980a, b), Lee and Wilde (1980), assume that each firm in R&D competition makes once-and-for-all expenditure which determines the winner.

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(ii) Reinganum (1981, 1982) consider a dynamic R&D race where each firm chooses a time path of expenditures. However, because the exponential distribution is used in these models, the knowledge acquired in the past does not change the probability of the current success in the race. As a result, the equilibrium strategies are independent of time.[1]

(iii) Harris and Vickers (1985) analyze a model of a race where the winner is the first player to reach the finish line. Fershtman and Rubinstein (1997) consider an interactive model in which two players search for a single hidden treasure in one of a given set of labeled boxes. In both models the players know the upper bound of the costs: the distance in the first model and the number of boxes in the second model.

In our paper, we combine and extend directions (ii) and (iii). In particular, we analyze a dynamic model where n players search for a treasure hidden somewhere on an island. Players in our model can observe what the other players have searched in all previous periods. As a result, each period, the size of the island shrinks to the previous-period size minus the part that has been searched by all players in the previous period.

The game we consider has Schelling's [2] "conflict of partnership and competition" property: players are naturally competing against each other each period, but each player benefits from the other players' previous periods unsuccessful search, because it increases his chance to find the treasure in the current period.

There are many examples of this situation: several detectives (police units) are looking for a criminal; several journalists are looking for a movie star in the city hotels; R&D race between several companies or several laboratories where search is an intensity of research.

We assume that only one player can obtain the treasure. If several players find the treasure simultaneously (search the same part of the island), each of them incurs costs but the treasure will be destroyed (players do not get any treasure). This assumption is standard in the R&D literature (see Chatterjee and Evans, 2004). It can be justified

[1] See Reinganum (1989) for more detail discussion about (i) and (ii).

[2] See Schelling (1971).

on the ground that if several players discover the treasure simultaneously, they will be involved in the fierce competition afterwards and run out of any surplus. A good example of this situation for just two players is 1960s Lockheed and Douglas jet development competition.[3] Many examples of simultaneous discoveries in science can be found in Merton (1973).

The main questions we answer in this paper include: how many periods does it take to discover the treasure? What are the equilibrium searching strategies (individual search plans)? Is it more efficient to search with more players? Will the treasure be found faster with more players?

The game we analyze is stochastic. We restrict our attention to individual Markov strategies. It not only simplifies the exposition considerably, but also allows to make a meaningful comparison between n-player case and 1-player case. We are able to characterize a symmetric Markov equilibrium for n players and find that

- any search with $n \geq 2$ players is *always* less efficient than one-player search;
- there are two sources of this inefficiency:

  (a) more players search too fast,

  (b) more players take not all efficient projects.

Our paper is related to the individual search literature; see Ross (1983) and Gittins (1989). However, players are assumed to search strategically in our model.

Chatterjee and Evans (2004) analyze R&D model which is similar to ours. They allow each of two firms to observe the other's past choices and search strategically. Their firms have to choose between two research projects. We have only one research project in their notation. Their model in a sense is complementary to our model. While agents in their model decide which area to search in (the size is predetermined), agents in our model decide how much area to search in (the location has no importance).

[3]For more detail see The Economist, 1985; and Chatterjee and Evans, 2004.

The paper is organized as follows. We start from an illustrative example where two players are looking for the treasure in Section 2. A general n-player model is developed and the main Bellman equation is described in section 3. We find the solution of the Bellman equation and discuss properties of this solution in Section 4. The two-player case allows the computation and detailed discussion of the equilibrium. This case is presented in Section 5. Section 6 concludes.

2 Example

Consider the following situation. Two pirates are searching for a hidden treasure on a beach line of length $x = 90$ miles. They start from the opposite sides. If the treasure is found, the game ends. The treasure's value is $R = \$100$ and it has equal chance to be at any place on the beach line. Each period t, the pirates know how much unexplored beach line left, $x(t)$, and simultaneously decide how much to search. The search is costly. In particular, pirate i has to pay $c = \$1$ per mile of search, or $cI^i(t)$ at period t. All future payoffs are discounted with a common discount factor of $\delta = 0.25$.[4] We assume (like in the Nash Demand game) that if the pirates decide to search together more than the remaining unexplored beach line, $I^1(t) + I^2(t) > x(t)$, the treasure is destroyed, the pirates have to pay the search costs, and the game ends.

In this section, we answer the following questions: What is a symmetric Markov perfect equilibrium? What is the maximal number of periods (the worst case scenario) when, in the symmetric Markov equilibrium, the pirates find the treasure for sure?

Given that the pirates discount future, it is intuitively clear that the treasure, if it is found, should be found in a finite number of periods. To make the exposition clear, as a start let us assume that the pirates can search only once. How much should each pirate search in a symmetric Markov equilibrium? Note that pirate 1's expected value from the search if he is allowed to search only once is

[4]One possible motivation for a discount factor is that there is a 75% chance that the game terminates at the end of each period.

$$V_1(x) = R\frac{I^1}{x} - cI^1 = \left(\frac{R}{x} - c\right) I^1,$$

where x is the length of the beach line, R is the treasure value, I^1 measures the mileage which pirate 1 has decided to search, c is the mileage cost.

It is evident that each pirate wants to search as much beach line as possible, if $R > cx$. Since in the example, $\$100 = R > cx = \$90$, in the symmetric Markov equilibrium each pirate searches a half of the beach line, $x/2$ miles. Therefore,

$$V_1(x) = \frac{R - cx}{2} = \frac{100 - x}{2}. \tag{1}$$

In particular,

$$V_1(90) = \frac{100 - 90}{2} = 5.$$

Now, suppose that the pirates can search for at most two periods. How much should each pirate search in the first period (if at all) and in the second period in a symmetric Markov perfect equilibrium? Pirate 1's expected value from the search if he is allowed to search for at most two periods is

$$V_2(x) = \left(R\frac{I^1}{x} - I^1\right) + \delta\left(1 - \frac{I^1 + I^2}{x}\right) V_1(x - I^1 - I^2),$$

where I^1 measures the mileage which pirate 1 has decided to search, the first bracket is the expected value of finding the treasure in the first period, and the second term is the expected value of finding the treasure in the second period. Note that if the treasure is not found in the first period, the unexplored beach line shrinks to $(x - I^1 - I^2)$ in the second period and the pirates obtain the expected value, $V_1(x - I^1 - I^2)$, from the beach line of that length.

Since the second period length $(x - I^1 - I^2)$ is never longer than x and $R > cx$ in the example, using (1), we get

$$V_2(x) = \left(R\frac{I^1}{x} - I^1\right) + \delta\left(1 - \frac{I^1 + I^2}{x}\right)\frac{R - x + I^1 + I^2}{2}.$$

The optimal search in the first period, I^1, satisfies the first order condition

$$\left(\frac{R}{x} - 1\right) + \delta\left(-\frac{1}{x}\right)\frac{R - x + I^1 + I^2}{2} + \delta\left(1 - \frac{I^1 + I^2}{x}\right)\frac{1}{2} = 0,$$

or

$$2\left(R - x\right) - \delta\left(R - x + I^1 + I^2\right) + \delta\left(x - I^1 - I^2\right) = 0.$$

In the symmetric equilibrium $I^1 = I^2$. Therefore, the previous equation becomes

$$2\left(R - x\right) - \delta\left(R - x + 2I^1\right) + \delta\left(x - 2I^1\right) = 0$$

and

$$I^1 = I^2 = \frac{2(R - x) - \delta(R - 2x)}{4\delta} = 2(100 - x) - 0.25(100 - 2x).$$

Hence, we obtain

$$I^1 = I^2 = \begin{cases} 175 - 1.5x, & \text{if} \quad 87.5 < x \leq 90, \\ x/2, & \text{if} \quad x \leq 87.5. \end{cases} \tag{2}$$

In particular, if $x = 90$,

$$I^1 = I^2 = 40.$$

Therefore,

$$V_2(x) = \begin{cases} \frac{1}{x}\left(-\frac{1}{2}(100 - x)^2 + \frac{50}{2}(100 - x) + \frac{625}{2}\right), & \text{if} \quad 87.5 < x \leq 90, \\ \frac{100 - x}{2}, & \text{if} \quad x \leq 87.5. \end{cases} \tag{3}$$

In particular,

$$V_2(90) = \frac{205}{36}.$$

Now, suppose that the pirates can search for at most three periods. How much should each pirate search in the first period (if at all), in the second period, and in the third period in a symmetric Markov perfect equilibrium? Pirate 1's expected value from the search if he is allowed to search for at most three periods is

$$V_3(x) = \left(R\frac{I^1}{x} - I^1\right) + \delta\left(1 - \frac{I^1 + I^2}{x}\right) V_2(x - I^1 - I^2),$$

where I^1 measures the mileage which pirate 1 has decided to search in the first period, the first bracket is the expected value of finding the treasure in the first period, and the second term is the expected value of finding the treasure after the first period. Note that if the treasure is not found in the first period, the unexplored beach line shrinks to $(x - I^1 - I^2)$ in the second period and the pirates obtain the expected value, $V_2(x - I^1 - I^2)$, from the beach line of that length.

Suppose that in equilibrium, $x - I^1 - I^2 > 87.5$, then using (3) and $R = 100$, we get

$$V_3(x) = \left(100\frac{I^1}{x} - I^1\right) + \frac{\delta}{x}\left(-\frac{1}{2}(100 - x + I^1 + I^2)^2 + 25(100 - x + I^1 + I^2) + \frac{625}{2}\right).$$

The optimal search in the first period, I^1, satisfies the first order condition

$$\left(\frac{100}{x} - 1\right) + \frac{\delta}{x}\left(-(100 - x + I^1 + I^2) + 25\right) = 0,$$

or

$$100 - x - \delta\left(75 - x + I^1 + I^2\right) = 0.$$

In the symmetric equilibrium $I^1 = I^2$. Therefore,

$$I^1 = I^2 = \frac{1}{2\delta}\left(100 - 75\delta - x(1-\delta)\right).$$

In particular, in the example for $x = 90$ and $\delta = .25$, we get

$$I^1 = I^2 = 2\left(100 - \frac{75}{4} - 90\frac{3}{4}\right).$$

$$I^1 = I^2 = \frac{55}{2}.$$

Since $x - I^1 - I^2 = 90 - 55 = 35 < 87.5$, we get a contradiction to the original assumption. That means the pirates find the treasure in at most two periods. Therefore, $V_3(x) = V_2(x)$ in the example.

Hence, each pirate should search 40 miles in the first period and, if the treasure is not found, another 5 miles in the second period in the Markov perfect equilibrium. It is always optimal to search for at most (in the worst case scenario) two periods even if pirates are allowed to search for *any* number of periods. The two-period search procedure we have described is a unique Markov perfect equilibrium of a game where pirates can search for any number of periods. This example illustrates an approach we will apply in a general model and demonstrates the optimal search structure.

3 The Model

n players are searching for a treasure which is hidden somewhere on an island. The value of the treasure is R for all players. Let $x(0) > 0$ and $x(t) \geq 0$ denote the initial and the current island size. The treasure has equal chances to be at any part of the island.[5] At period $t \geq 0$, player i ($i = 1, 2, ..., n$) knows the *history* $h(t) = (x(0); I(0), ..., I(t-1))$ (where $I(k) = (I^1(k), ..., I^n(k)), k = 0, ..., t-1$) and chooses how much to search $I^i(t)$ (where $0 \leq I(t) \leq x(t)$).

[5] We focus our attention on uniform distribution because this is the most realistic situation when there is no information about the island. Note that this is the worst situation from the players point of view.

If $I^1(t) + ... + I^n(t) > x(t)$, the game ends and each player i gets payoff

$$-(I^i(0) + \delta I^i(1) + ... + \delta^t I^i(t)),$$

where δ is the common discount factor.

If $I^1(t) + ... + I^n(t) = x(t)$, the game ends and each player i has a $\frac{I^i(t)}{x(t)}$ chance to find the treasure. The expected payoff of player i in this case is

$$\delta^t \frac{I^i(t)}{x(t)} R - (I^i(0) + \delta I^i(1) + ... + \delta^t I^i(t)).$$

If $I^1(t) + ... + I^n(t) < x(t)$, each player i has a $\frac{I^i(t)}{x(t)}$ chance to find the treasure and the game ends with probability $\frac{I^1(t)+...+I^n(t)}{x(t)}$.

We assume that a player can see how much the other players have searched so far before making searching plans for the next period. If the treasure is not found in period t, the island size shrinks into $x(t+1) = x(t) - (I^1(t) + ... + I^n(t))$. New size of the island is equal to the previous island size minus the part that has been searched. Note that all investments are forfeit, but only one player (if any) can find the treasure.

Player i's strategy is an infinite sequence of functions specifying an investment at each period contingent upon any possible sequence of previous investments. However, the game we consider is stochastic and any history can be "summarized" by the "state" - the current size of the island. The current size of the island follows a Markov process; that is the probability distribution on the next period state is determined by the current state and the current investments. We will restrict our attention only to Markov strategies in which the past influences the current play only through its effect on the current island size. A pure Markov strategy for player i is a time-invariant map $I^i : X \to X$, where $X \in [0, x(0)]$. Therefore, the solution concept is a *Markov perfect equilibrium* (MPE).

Player i has to solve the following Bellman equation:

$$V^i(x) = \max_{0 \le I^i \le x - I^{-i}} \{-I^i + RI^i/x + \delta(1 - (I^i + I^{-i})/x)V^i(x - I^i - I^{-i})\}, \quad (4)$$

where the first term in this equation describes player's costs of investment. The second term is player's expected value from finding the treasure in the current period. The last term is player's expected value from winning in the future periods.

4 Analysis of the Model

Define the part of the island which player i does not search in the current period by

$$y := x - I^i$$

and the part of the island which both players do not search in the current period (the remaining part of the island) by

$$z := x - I^i - I^{-i}.$$

Equation (4) can be rewritten in the following way

$$V^i(x) = \max_{I^{-i} \le y \le x} \{-(x - y) + R(x - y)/x + \delta z V^i(z)/x\}. \quad (5)$$

Note, x, y, z, R and $V(x)$ are of the same unit measure. For convenience, we make the following substitution

$$x := x/R, \;\; y := y/R, \;\; V := V/R, \;\; R := 1; \quad (6)$$

to work with unit free variables. Equation (5) transforms to

$$V^i(x) = \max_{I^{-i} \le y \le x} \{-(x - y) + (x - y)/x + \delta z V^i(z)/x\}. \quad (7)$$

Let us derive player's value of the game, $V^i(x)$. To simplify exposition it is convenient to introduce function

$$\Psi^i(x) := xV^i(x). \tag{8}$$

From definition (8) it follows that

$$\Psi^i(x) \geq 0 \text{ for any x.} \tag{9}$$

Note that in the symmetric equilibrium, $I^i = I$ and $I^{-i} = (n-1)I$. Equation (7) in terms of $\Psi(x)$ can be rewritten as

$$\Psi^i(x) = \max_{(n-1)I \leq y \leq x} \{(1-x)(x-y) + \delta\Psi^i(z)\} =: B\Psi^i(z). \tag{10}$$

The following Lemma follows from the contraction mapping theorem, see for example Stokey, Lucas and Prescott (1989).

Lemma 1. *If $\delta < 1$, the operator on the right hand side of equation (10) is a contraction operator. Therefore, equation (10) has a unique solution, Ψ^i, that can be obtained as the limit of the following sequence $\{\Psi^i_k\}$, where*

$$\Psi^i_0 \equiv 0, \qquad \Psi^i_k := B\Psi^i_{k-1} \quad k = 1, 2, ... \tag{11}$$

4.1 Construction of sequences $\{\Psi^i_k\}$ and $\{V^i_k\}$

Note that with the help of Lemma 1 we can construct sequence $\{\Psi^i_k\}$. The procedure is equivalent to using the backward induction argument.

4.1.1 Construction of Ψ_1 and V_1.

Let us start from the end of the investment process. What will be the value of the game, if players could only search the whole island in at most one period? Equation (10) transforms to

$$\Psi_1(x) = \max_{(n-1)I \leq y \leq x} \{(1-x)(x-y)\}. \tag{12}$$

Since duplication is not feasible players will invest together $I^i + I^{-i} = x$ in the equilibrium. Consequently, in the symmetric equilibrium

$$y = \frac{(n-1)x}{n}, \text{ and } z = 0. \tag{13}$$

Problem (12) has the following solution

$$\Psi_1(x) = \begin{cases} x(1-x)/n, & \text{if } x < u_1 = 1, \\ 0, & \text{if } x \geq u_1 = 1, \end{cases} \tag{14}$$

where u_1 is a positive root of polynomial $x(1-x)/n$, that is $u_1 = 1$. For a future reference note that

$$x(1-x)/n = -\frac{1}{n}(1-x)^2 + \frac{1}{n}(1-x). \tag{15}$$

If the players can search the island in at most one period, then the symmetric MPE is

$$y(x) = \begin{cases} \frac{(n-1)x}{n}, & \text{if } x < u_1, \\ x, & \text{if } x \geq u_1. \end{cases} \tag{16}$$

The optimal first-period searching is independent from the discount factor because there is no delay.

Define $V_k(x) := \Psi_k(x)/x$, for any $x \geq 0$. V can be obtained as the limit of sequence $\{V_k\}$. From the above definition it follows

$$V_1(x) = \begin{cases} (1-x)/n, & \text{if } x < u_1, \\ 0, & \text{if } x \geq u_1. \end{cases} \tag{17}$$

4.1.2 Construction of Ψ_2 and V_2.

What will be the value of the game, if players can search the whole island in at most two periods? In general there could be three possibilities depending on island size. The first possibility is that the players search the whole island in just one period. Intuitively this happens for small values of x because it is too costly to wait for another period when the island is very small. The second possibility is that the players finish the island in two periods. This happens for middle values of x. Finally, players can find searching to be too costly and don't search at all. This happens when costs are very large.

Now let us formally construct $\Psi_2(x)$. Equation (10) in this case transforms to

$$\Psi_2(x) = \max_{(n-1)I \leq y \leq x} \{(1-x)(x-y) + \delta\Psi_1(z)\}. \tag{18}$$

Necessary condition for y to be the optimal value in the interior of $[0, x]$ is

$$-(1-x) + \delta\Psi_1'(z) = 0. \tag{19}$$

From condition (19) and expression (14), it follows

$$-(1-x) + \delta\left(\frac{1-2z}{n}\right) = 0.$$

Consequently,

$$z(x) = \frac{n(x-1)+\delta}{2\delta}. \tag{20}$$

Therefore

$$y = \frac{x(n-1)+z(x)}{n} = \frac{2\delta(n-1)x + n(x-1)+\delta}{2\delta n}. \tag{21}$$

Plugging (21) into equation (18), we obtain

$$\Psi_2(x) = \begin{cases} \Psi_1(x), & \text{if} \quad x < t_1, \\ -\frac{4\delta-2n+n^2}{4\delta n}(1-x)^2 + \frac{1}{2n}(1-x) + \frac{\delta}{4n}, & \text{if} \quad t_1 \leq x < u_2, \\ 0, & \text{if} \quad x \geq u_2, \end{cases} \tag{22}$$

where u_2 is the largest root of polynomial $-\frac{4\delta-2n+n^2}{4\delta n}(1-x)^2+\frac{1}{2n}(1-x)+\frac{\delta}{4n}$, specifically $u_2 = 1 + \frac{\sqrt{4\delta(s+1)+1}-1}{4(s+1)}$ where

$$s = \frac{n(n-2)}{4\delta}. \tag{23}$$

t_1 is such island size that players are indifferent between searching the island in two periods, or in one period:

$$\Psi_1(t_1) = \Psi_2(t_1). \tag{24}$$

From (15) and (22) we find

$$t_1 = 1 - \frac{\delta}{n}. \tag{25}$$

If the players can search the island in at most two periods, the symmetric equilibrium is

$$y(x) = \begin{cases} \frac{(n-1)x}{n}, & \text{if} \quad x < t_1, \\ \frac{2\delta(n-1)x+n(x-1)+\delta}{2\delta n}, & \text{if} \quad t_1 \le x < u_2, \\ x, & \text{if} \quad x \ge u_2 \end{cases} \tag{26}$$

and the value function is

$$V_2(x) = \begin{cases} V_1(x), & \text{if} \quad x < t_1, \\ \Psi_2(x)/x, & \text{if} \quad t_1 \le x < u_2, \\ 0, & \text{if} \quad x \ge u_2. \end{cases} \tag{27}$$

4.1.3 Construction of Ψ_k and V_k.

Consider $\Psi_2(x)$ and $\Psi_1(x)$. Note that they both are polynomials in the following form:

$$\Psi_k(x) = \frac{a_k(1-x)^2 + b_k(1-x) + c_k}{n}, \quad k = 1, 2, \tag{28}$$

where

$$a_1 = -1, \quad b_1 = 1, \quad c_1 = 0;$$

and

$$a_2 = -1 - s, \quad b_2 = \frac{1}{2}, \quad c_2 = \frac{\delta}{4}.$$

From (10),

$$\Psi_k(x) = \max_{(n-1)I \le y \le x} \{(1-x)(x-y) + \delta \Psi_{k-1}(z)\} =: B\Psi_{k-1}(z). \tag{29}$$

One can note that if function $\Psi_{k-1}(x)$ in (29) is a quadratic polynomial, $\Psi_k(x) = B\Psi_{k-1}$ has to be a quadratic polynomial as well. Let us find a_k, b_k, and c_k for any k.

Necessary condition for y to be the optimal value in the interior of $[0, x]$ is

$$(1-x) = \delta \Psi'_{k-1}(z). \tag{30}$$

Therefore from (28) and (30),

$$z(x) = 1 + \frac{\delta b_{k-1} + (1-x)n}{2\delta a_{k-1}}. \tag{31}$$

Next we find

$$y = x + \frac{z-x}{n} = x + \frac{(1-x)(n + 2a_{k-1}\delta)}{2na_{k-1}\delta} + \frac{b_{k-1}}{2na_{k-1}}. \tag{32}$$

Hence

$$\Psi_k(x) = -(1-x)\left(\frac{(1-x)(n+2a_{k-1}\delta)}{2na_{k-1}\delta} + \frac{b_{k-1}}{2na_{k-1}}\right) + \delta\Psi_{k-1}(z). \tag{33}$$

Finally, we have

$$a_k = -1 + \frac{s}{a_{k-1}}, \quad b_k = -\frac{b_{k-1}}{2a_{k-1}}, \quad c_k = \delta\left(c_{k-1} - \frac{b_{k-1}^2}{4a_{k-1}}\right). \tag{34}$$

Result 1. *The system of difference equations (34) has the following solution*

$$a_k = \frac{\left(\frac{\sqrt{1+4s}-1}{2}\right)^{k+1} - \left(\frac{-\sqrt{1+4s}-1}{2}\right)^{k+1}}{\left(\frac{\sqrt{1+4s}-1}{2}\right)^{k} - \left(\frac{-\sqrt{1+4s}-1}{2}\right)^{k}}, \ \ b_k = \frac{(-\frac{1}{2})^{k-1}\sqrt{1+4s}}{\left(\frac{\sqrt{1+4s}-1}{2}\right)^{k} - \left(\frac{-\sqrt{1+4s}-1}{2}\right)^{k}}, \tag{35}$$

$$c_k = \sum_{i=2}^{k} \frac{-(1+4s)\delta^k}{(4\delta)^{i-1}\left[\left(\frac{\sqrt{1+4s}-1}{2}\right)^{i} - \left(\frac{-\sqrt{1+4s}-1}{2}\right)^{i}\right]\left[\left(\frac{\sqrt{1+4s}-1}{2}\right)^{i-1} - \left(\frac{-\sqrt{1+4s}-1}{2}\right)^{i-1}\right]}.$$

Proof. a_k, b_k and c_k are derived in the Appendix.

Corollary 1. *If $n = 2$ the system of difference equations (34) has the following solution*

$$a_k = -1, \ \ b_k = \frac{1}{2^{k-1}}, \ \ c_k = \left(\frac{(4\delta)^{k-1} - 1}{(4\delta - 1)4^{k-1}}\right)\delta. \tag{36}$$

Define t_k to be the island size for which players are indifferent between searching the island in k or in $k+1$, i.e.

$$\Psi_k(t_k) = \Psi_{k+1}(t_k). \tag{37}$$

In addition define u_k to be the minimum value where $\Psi_k(x) = 0$. We derive the following results for two separate cases $n = 2$ and $n > 2$.

Result 2. *If $n = 2$ the players' indifference points are*

$$t_k = 1 + \frac{c_{k+1} - c_k}{b_{k+1} - b_k}. \tag{38}$$

$\Psi_k(x)$ is strictly positive for any $0 < x < u_k$ and zero for any $x \geq u_k$, where

$$u_k = 1 - b_k + \sqrt{b_k^2 + 2c_k}. \tag{39}$$

Proof. It follows from equations (37) and (28).

Result 3. *If $n > 2$ the players' indifference points are*

$$t_k = 1 + \frac{b_{k+1} - b_k - \sqrt{(b_{k+1} - b_k)^2 - 4(a_{k+1} - a_k)(c_{k+1} - c_k)}}{2(a_{k+1} - a_k)}. \tag{40}$$

$\Psi_k(x)$ is strictly positive for any $0 < x < u_k$ and zero for any $x \geq u_k$, where

$$u_k = 1 + \frac{b_k - \sqrt{b_k^2 - 4a_k c_k}}{2a_k}. \tag{41}$$

Proof. It follows from equations (37) and (28).

If the players can search the island in at most k periods, in the symmetric equilibrium

$$\Psi_k(x) = \begin{cases} \Psi_{k-1}(x), & \text{if} \quad x < t_{k-1}, \\ \frac{a_k(1-x)^2 + b_k(1-x) + c_k}{n}, & \text{if} \quad t_{k-1} \leq x < u_k, \\ 0, & \text{if} \quad x \geq u_k, \end{cases} \tag{42}$$

and the value function is

$$V_k(x) = \begin{cases} V_{k-1}(x), & \text{if} \quad x < t_{k-1}, \\ \Psi_k(x)/x, & \text{if} \quad t_{k-1} \leq x < u_k, \\ 0, & \text{if} \quad x \geq u_k. \end{cases} \tag{43}$$

Thus $V_k(x)$ is described by (43) with coefficients given by (34) and points of regime change given by (38), (39), (40) and (41). The value function in the case when the agent can invest not more than 2 times and $\delta = 0.25$ is plotted in Figure 1.

Now let us address the question of what is the minimum k that $V(x) \equiv V_k(x)$. In other words what is the maximum number of periods the search will be finished for any value of x. Let us define $\delta_k(n)$ to be the root of the following equation [6]

$$\delta_k(n) \quad : \quad u_{k-1}(n) = t_{k-1}(n) \quad k = 3, 4, ... \tag{44}$$

[6] Note that from (14) and (25) it follows that $u_1 > t_1$, that is the minimum number of periods is always greater than 1.

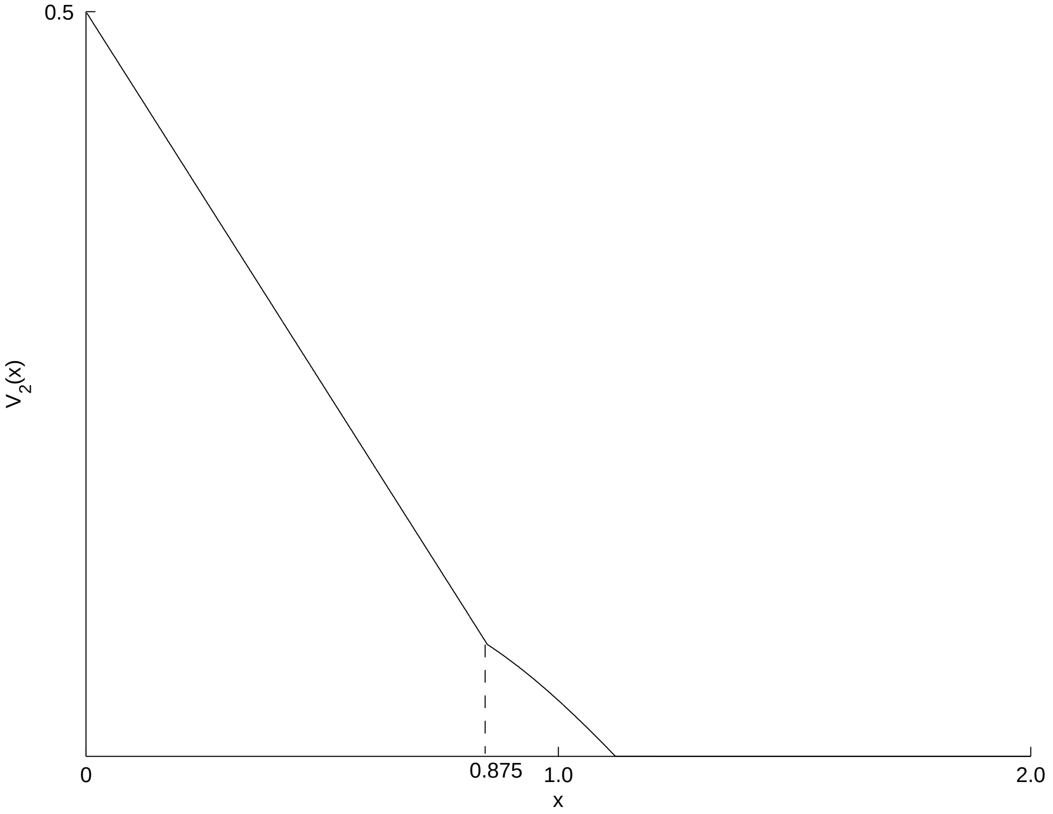

Figure 1: $V_2(x)$

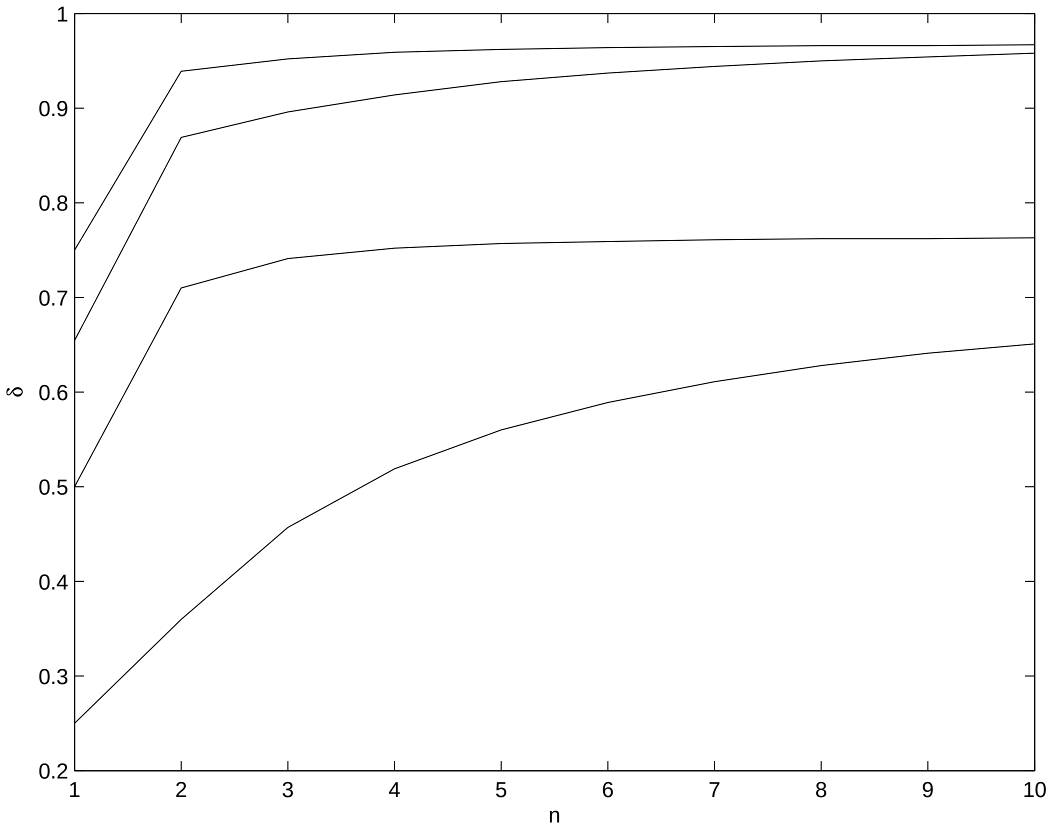

Figure 2: Different zones

The following theorem was proved in Belenky, Smirnov and Wait (2007) for the case when there is only one player

Theorem 1. *If $n = 1$, equation (44) has the following unique solution*

$$\delta_k = \cos^2 \frac{\pi}{k} \qquad k = 3, 4, \dots \tag{45}$$

With the help of (44) we lay out the parametric space (n, δ) into zones, see picture 2.

4.2 Properties

Inefficiency of $n > 1$ players in comparison with $n = 1$:

(1)faster search

(2)less projects

The maximal number of periods in order to finish the project as a function of delta and N.

Example: compare n = 2 and n = 5.

Two reasons for less delay with more players:

(1) More zones in n = 2,

(2) The "transition" points move.

5 Conclusion

In this paper, dynamic search model is analyzed. We suggest a new modeling approach when the value of the prize (treasure) is known at the beginning of the game, but the search cost is not known in advance. The resulting game is stochastic and we consider individual Markov strategies.

We find a symmetric Markov Perfect equilibrium and analyze its properties. Markov Perfect equilibrium allows us to make a comparison between one-player case and more-than-one-player case. Belenky, Smirnov, and Wait (2006) consider one-player case and find the optimal search strategy. We demonstrate that if the number of players increases, then the efficiency goes down. There are two reasons for that. First, players face more competition and as the result they search faster (over smaller number of periods) than optimal. Second, players do not search if the initial searching area is (too) big, even though it would be optimal to search this area in the one-player case.

We are able to derive the maximal number of periods (the worst case scenario) which is required (in the equilibrium) to find the treasure as a function of the number of players and the discount factor. We present our finding as a graph (see Figure 2) where

- for each number of players the worst- (the longest) and the best- (the fastest) case scenario can be found (see different zones);

- for each discount factor the worst- (the longest) and the best- (the fastest) case scenario can be found (see different zones).

There are many applications for the game we developed. It seems to us that the most interesting application is to R&D and patent races. As we indicate in the Introduction our approach brings a new light into the existing literature. Unlike the existing literature, we derive an analytical symmetric Markov perfect equilibrium. The equilibrium is transparent: the whole plan of actions is described.

Appendix

Proof of result 1

Derivation of a_k, b_k and c_k

Using

$$a_k = -1 + \frac{s}{a_{k-1}}, \quad b_k = -\frac{b_{k-1}}{2a_{k-1}}, \quad c_k = \delta\left(c_{k-1} - \frac{b_{k-1}^2}{4a_{k-1}}\right) \tag{34}$$

let us show

$$a_k = \frac{\left(\frac{\sqrt{1+4s}-1}{2}\right)^{k+1} - \left(\frac{-\sqrt{1+4s}-1}{2}\right)^{k+1}}{\left(\frac{\sqrt{1+4s}-1}{2}\right)^{k} - \left(\frac{-\sqrt{1+4s}-1}{2}\right)^{k}}, \quad b_k = \frac{(-\frac{1}{2})^{k-1}\sqrt{1+4s}}{\left(\frac{\sqrt{1+4s}-1}{2}\right)^{k} - \left(\frac{-\sqrt{1+4s}-1}{2}\right)^{k}}, \tag{35}$$

$$c_k = \sum_{i=2}^{k} \frac{-(1+4s)\delta^k}{(4\delta)^{i-1}\left[\left(\frac{\sqrt{1+4s}-1}{2}\right)^{i} - \left(\frac{-\sqrt{1+4s}-1}{2}\right)^{i}\right]\left[\left(\frac{\sqrt{1+4s}-1}{2}\right)^{i-1} - \left(\frac{-\sqrt{1+4s}-1}{2}\right)^{i-1}\right]}.$$

Define

$$P_k := \prod_{j=1}^{k} a_j \qquad k = 1, 2, \dots \tag{46}$$

Using (34) we get the following second-order difference equation

$$P_k + P_{k-1} - sP_{k-2} = 0 \quad k \geq 2. \tag{47}$$

The initial conditions are $P_1 = -1$ and $P_2 := 1 + s$. The characteristic equation $z^2 + z - s = 0$ has two real roots

$$z_1 = \frac{\sqrt{1+4s} - 1}{2}, \quad z_2 = \frac{-\sqrt{1+4s} - 1}{2}. \tag{48}$$

Write solutions to equation (47) in form $P_k = Az_1^{k+1} - Bz_2^{k+1}$ and use initial conditions to get $A = B = \frac{1}{\sqrt{1+4s}}$. Next, use $a_k = \frac{P_k}{P_{k-1}}$ to derive a_k in (35).

Note that from (34) it follows that $a_{k-1} = -\frac{b_{k-1}}{2b_k} = \frac{P_{k-1}}{P_{k-2}}$. One can see that b_k is an inverse of P_k and consequently can easily guess b_k in (35).

c_k has to satisfy the initial condition $c_1 = 0$. Introduce $d_k = \frac{c_k}{\delta^k}$ and rewrite the difference equation as

$$d_k = d_{k-1} + \frac{b_k b_{k-1}}{2}.$$

Substitute the initial condition $d_1 = 0$ to derive

$$d_k = d_1 + \sum_{i=2}^{k}(d_i - d_{i-1}) = \sum_{i=2}^{k} \frac{b_k b_{k-1}}{2}.$$

Finally we have

$$c_k = \delta^k d_k = \delta^k \sum_{i=2}^{k} \frac{b_k b_{k-1}}{2}.$$

Substitute b_k from (35) to derive c_k in (35).

References

[1] Anat Admati, and Motty Perry (1991): "Joint Projects without Commitment," *Review of Economic Studies*, vol. 58, no. 2, 259-276.

[2] Vitali Belenky, Vladimir Smirnov, and Andrew Wait (2006): "Reaching for the prize: investment with uncertain costs," *Mimeo,* University of Sydney.

[3] Kalayan Chatterjee, and Robert Evans (2004): "Rivals Search for Buried Treasure: Competition and Duplication in R&D," *Rand Journal of Economics,* 35, 1, 160-183.

[4] Dasgupta and Stiglitz (1980a): "Industrial Structure and the Nature of Innovative Activity," *The Economic Journal*, 90, 266-293.

[5] Dasgupta and Stiglitz (1980b): "Uncertainty, Industrial Structure and the Speed of R and D," *Bell Journal of Economics*, 11, 1-28.

[6] The Economist (1985): "Aircraft Industry: The Big Six: A Survey," June 1, 8.

[7] Gene Grossman, and Carl Shapiro (1986): "Optimal dynamic P&D programs," *Rand Journal of Economics,* 17, 4, 581-593.

[8] Chaim Fershtman and Ariel Rubinstein (1997): "A Simple Model of Equilibrium in Search Procedures," *Journal of Economic Theory,* 72, 432-441.

[9] J. Gittins (1989): *Multi-armed Bandit Allocation Indices*. New York: Wiley.

[10] Harris and Vickers (1985) "Perfect equilibrium in a model of a race," *Review of Economics Studies,* 52:193-209.

[11] Lee and Wilde (1980) "Market structure and innovation: A reformulation," *Quaterly Journal of Economics,* 94: 429-436.

[12] Loury (1979) "Market structure and innovation," *Quaterly Journal of Economics,* 93:395-410.

[13] Alexander Matros, and Vladimir Smirnov (2005): "Investment Game with Uncertain Costs," *Mimeo,* University of Sydney.

[14] R. Merton (1973): "Singletons and Multiples in Scientific Discovery." In R. Merton and N. Storer, eds. *The Sociology of Science.* Chicago: University of Chicago Press.

[15] D. Neher (1999): "Staged Financing: An Agency Perspective," *Review of Economic Studies*, Vol. 66, no. 2, 255-74.

[16] Reinganum (1981): "Dynamic Games of Innovation," *Journal of Economic Theory,* 25, 21-41.

[17] Reinganum (1982): "A Dynamic Game of R and D: Patent Protection and Competitive Behavior," *Econometrica,* vol 50, no. 3, 671-688.

[18] Reinganum (1989): "The Timing of Innovation: Research, Development, and Diffusion," *Handbook of Industrial Organization*, Vol. 1, 849-908.

[19] S. Ross (1983): *Introduction to Stochastic Dynamic Programming.* New York: Academic Press.

[20] Schelling (1971): "Dynamic Models of Segregation," *Journal of Mathematical Sociology* 1, 143-186.

[21] N. Stokey, R. Lucas and E. Prescott (1989) *Recursive Methods in Economic Dynamics*. Harvard University Press.